Uniqueness in Law for Stochastic Boundary Value Problems

Anna Capietto * & Enrico Priola †

Dipartimento di Matematica, Università di Torino
Via Carlo Alberto 10, 10123 Torino, Italy.

October 15, 2009

Mathematics Subject Classification (2000): 60H10, 34F05, 60H07.

Key words: stochastic boundary value problems, anticipative Girsanov theorem, uniqueness in law.

Abstract: We study existence and uniqueness of solutions for second order ordinary stochastic differential equations with Dirichlet boundary conditions on a given interval. In the first part of the paper we provide sufficient conditions to ensure pathwise uniqueness, extending some known results. In the second part we show sufficient conditions to have the weaker concept of uniqueness in law and provide a significant example. Such conditions involve a linearized equation and are of different type with respect to the ones which are usually imposed to study pathwise uniqueness. This seems to be the first paper which deals with uniqueness in law for (anticipating) stochastic boundary value problems. We mainly use functional analytic tools and some concepts of Malliavin Calculus.

1 Introduction

The object of this paper is the stochastic ordinary differential equation

$$\frac{d^2X_t}{dt^2} + f\Big(t, X_t, \frac{dX_t}{dt}\Big) = \frac{dW_t}{dt}, \qquad t \in [0,1], \tag{1.1}$$

subject to the boundary condition $X_0 = 0 = X_1$, where $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is a given continuous function and (W_t) is a one-dimensional Wiener process starting from 0 (note that $X_t = X(t)$, $t \in [0,1]$).

There is a wide literature on (anticipating) stochastic boundary value problems (see, for instance, [2], [7], [8], [16], [17], [18]). Methods for numerically solving stochastic boundary value problems are investigated as well (see [1] and the references therein). Usually, once the existence of a solution $X = (X_t)$ is guaranteed, the question of uniqueness is tackled in the *pathwise* sense (i.e., if Z is another solution to (1.1), then $X = Z$, $\mathbb{P}$-a.s.,

---

*e-mail: anna.capietto@unito.it. Supported by the M.I.U.R. research project 2007 "Topological and Variational Methods in the Study of Nonlinear Phenomena".

†e-mail: enrico.priola@unito.it. Supported by the M.I.U.R. research project 2006 "Kolmogorov Equations".

where $\mathbb{P}$ denotes the Wiener measure on $C_0([0,1])$, see Section 2). Having in mind an application of the contraction principle, it is usually required, roughly speaking when $f(t,x,y) = f(x)$, that f is globally Lipschitz with a Lipschitz constant small enough or that f satisfies a kind of monotonicity condition.

Our contribution is two-fold. On one hand, concerning pathwise uniqueness, we show in Section 3 that some of the methods of nonlinear analysis (see the seminal work [13] and the book [6]) for deterministic ordinary differential equations are suitable for improving some of the results already available in the literature. On the other hand, we propose a new step in the study of stochastic BVPs, i.e. we provide sufficient conditions for the weaker concept of uniqueness *in law* of solutions (i.e., if Z is another solution to (1.1), then $\mathbb{P}(X^{-1}(A)) = \mathbb{P}(Z^{-1}(A))$, for any Borel set $A \subset C_0([0,1])$). Such conditions are of different type w.r.t. the available results on pathwise uniqueness (see, in particular Section 4.5 and also Section 4.6, which contains a significant example). Roughly speaking, our Theorem 4.23 in Section 4.6 shows that uniqueness in law holds even if a "typical" non-resonance condition is violated on a discrete set of points. On the other hand, we do not know if pathwise uniqueness holds in such a case, since the usual methods of nonlinear analysis fail.

Note that, to the authors' knowledge, up to now uniqueness in law has been treated only for the well-studied (non-anticipating) Cauchy problem for stochastic differential equations (cf., for instance, [11]).

We first concentrate on a precise definition of the notion of solution. Indeed, according to the paper [16] by Nualart-Pardoux (which was the starting point of our research), we understand (1.1) in the integral sense, i.e., we require that $X : C_0([0,1]) \to C_0([0,1])$ is Borel measurable and that (setting $\frac{dX_t}{dt} = X_t'$)

$$X_t'(\omega) + \int_0^t f(s, X_s(\omega), X_s'(\omega))ds = X_0'(\omega) + \omega_t, \qquad t \in [0,1], \tag{1.2}$$

for any $\omega \in C_0([0,1])$, $\mathbb{P}$-a.s. (see Section 2 for the precise definition). Then existence and pathwise uniqueness of solutions to (1.2) are investigated, arguing for a fixed $\omega \in C_0([0,1])$. In Section 3.1 we use the global implicit function theorem and provide an existence and uniqueness result (Theorem 3.2) under a non-resonance type condition; this goal is reached after writing (1.2) as an abstract equation involving the Green's function of $-d^2/dt^2$ (with Dirichlet boundary condition). In Section 3.2 we give sufficient conditions (of Lipschitz type) on $f(t,x,y)$ which enable us to study the BVP (1.1) as a fixed point problem and to apply the contraction mapping principle. In particular, Corollaries 3.9 and 3.11 improve related results in [16, Section 1]. Section 3 ends with a discussion on the Fredholm alternative for (1.1).

Once this first aspect has been developed, it is quite natural to consider the case in which pathwise uniqueness is not guaranteed (see Section 4). To this purpose, we deal with the mapping $T : C_0([0,1]) \to C_0([0,1])$ introduced in [16]:

$$T_t(\omega) = \omega_t + \int_0^t f(s, Y_s(\omega), Y_s'(\omega))ds, \;\; \omega \in C_0([0,1]), \; t \in [0,1],$$

where $Y = (Y_t)$ is the solution to (1.1) corresponding to $f = 0$. In [16] it is shown that if T is *bijective* then existence and pathwise uniqueness hold for (1.1) (see also Proposition 2.4). We first show that even if T is not bijective, there always exists a measurable *left inverse S* of T provided that a solution X exists (see Lemma 4.13). This was our starting point to study uniqueness in law. Indeed, once the existence of a

left inverse is proved the aim is to use a non-adapted version of the Girsanov theorem recently proved by Üstünel-Zakai in [25] (see Section 4.2).
Remark that to study uniqueness in law we can not use the well known non-adapted version of the Girsanov theorem due to Ramer and Kusuoka (see [12], [20], and also [14, Section 4.1]). This result has been already applied to stochastic BVPs in [7], [8] and [16], in order to investigate the Markov property when a unique solution exists. The Ramer-Kusuoka theorem would require that T *is bijective* (i.e., pathwise uniqueness holds for (1.1)). This is not the case for the Girsanov theorem in [25] which, however, requires some additional hypotheses (involving Malliavin Calculus) which are not present in [12] and [20].
Although the formulation of [25, Theorem 3.3] involves Sobolev spaces of Malliavin Calculus, we find more useful to deal with the strictly related notion of H-differentiability (cf. Section 4.1 and see [14, Section 4.1.3] and [22]). By using the inverse function theorem and some functional analytic tools, we first show the H-differentiability of the transformation $F : \Omega \to \Omega$,

$$F_t(\omega) = -\int_0^t f(s, X_s(\omega), X_s'(\omega))ds, \ \ \omega \in \Omega, \ t \in [0,1],$$

where X is a given solution (see Theorem 4.14); it turns out that $S = I + F$ is the above mentioned left inverse of T. Then we prove an exponential estimate for the Skorohod integral of F (see Section 4.4) which is required in the Girsanov theorem of [25]. Remark that the known exponential estimates (cf. [23] and [25, Appendix B.8]) are not applicable to get our bound. We obtain the required exponential integrability assuming that f is bounded.
In Section 4.5 we prove a uniqueness in law result in the following form (assume for simplicity that $f(t,x,y) = f(x)$). If $f \in C_b^2(\mathbb{R})$, then uniqueness in law for (1.1) holds among all the solutions X such that the corresponding linearized equations

$$u_t'' + a_t(\omega)u_t = 0, \ \ u_0 = u_1 = 0,$$

where $a_t(\omega) = f'(X_t(\omega))$, $t \in [0,1]$, have the only solution $u = 0$, for any $\omega \in \Omega$, $\mathbb{P}$-a.s.. This means that uniqueness in law holds for (1.1) whenever one is able to prove that all solutions X to (1.1) verify our assumption on the linearized equation. In Section 4.6 we show a concrete class of BVPs for which this is possible. Note that in Theorem 4.23 of Section 4.6 we also establish existence of solutions; this is quite involved (see also Remark 4.24 where a more general existence result is formulated).
The previous condition on the linearized equation can be, roughly speaking, interpreted (from the nonlinear analysis point of view) as a requirement on the invertibility of the differential of the map S; indeed, as it is explained in the proof of Theorem 4.14, it ensures that S is a local homeomorphism. In order to obtain a global homeomorphism, and thus *pathwise* uniqueness, Section 3 shows that some additional assumptions (such as the non-resonance condition (3.6)) have to be added. Thus, a rough comparison between our pathwise and "in law" uniqueness results may be proposed in the sense that the fact that S is a local diffeomorphism is sufficient to guarantee uniqueness in law.
Finally, in Section 5 we tackle a problem which arises when dealing with non-adapted versions of the Girsanov theorem. It consists of the determination of an explicit expression for a Carleman-Fredholm determinant related to the mapping T (see (4.31)) This expression is reached in [7] and [16] with an involved proof based on Malliavin calculus.

We propose an alternative shorter proof based on a functional-analytic approach taken from the book [10]. We believe that this method can be extended to other situations in which the computation of Carleman-Fredholm determinants is of interest. We also use the methods of [10, Chapter XIII] to find the expression of the Malliavin derivative of F (see Proposition 4.16). An account of the ideas from [10] can be found in Appendix B.

Acknowledgments The authors are grateful to Paolo Cermelli for many helpful and fruitful discussions. They also thank the collegues of the I.N.D.A.M. project "Does noise simplify or complicate the dynamics of nonlinear systems?" for useful conversations.

Basic Notations $\Omega = C_0([0,1])$ denotes the Banach space of all real continuous functions on $[0,1]$ which vanish in $t=0$, endowed with the supremum norm $\|\cdot\|_0$. Moreover, $\mathcal{F}$ is the Borel σ-algebra on Ω and $\mathbb{P}$ the Wiener measure on Ω; $\mathbb{P}$ can be uniquely characterized by saying that on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the stochastic (coordinate) process $W = (W_t)$,

$$W_t(\omega) = \omega(t), \quad \omega \in \Omega,\ t \in [0,1], \tag{1.3}$$

is a real Wiener process (up to time $t=1$). As usual, when a property concerning Ω holds for any $\omega \in \Omega_0$, with $\Omega_0 \in \mathcal{F}$ and $\mathbb{P}(\Omega_0) = 1$, we say that this property holds $\mathbb{P}$-a.s.. The subspace C_0^1 of Ω consists of all C^1-functions vanishing at $t=0$ and $t=1$.
Let H_1 and H_2 be real separable Hilbert spaces (with inner product $\langle\cdot,\cdot\rangle_{H_k}$ and norm $|\cdot|_{H_k}$, $k=1,2$). A linear and bounded operator $L : H_1 \to H_2$ is said to be a Hilbert-Schmidt operator if for some orthonormal basis (e_n) in H_1 we have $\sum_{n\geq 1} |Le_n|_{H_2}^2 < \infty$. The space of all Hilbert-Schmidt operators will be indicated with $H_1 \otimes H_2$ or $\mathcal{HS}(H_1, H_2)$; it is a Hilbert space with the inner product $\langle\cdot,\cdot\rangle_{H_1\otimes H_2}$, $\langle R, S\rangle_{H_1\otimes H_2} = \sum_{n\geq 1} \langle Re_n, Se_n\rangle_{H_2}$ (see, for instance, [10, Chapter IV] or [21, Chapter VI]).
The corresponding Hilbert-Schmidt norm is indicated by $\|\cdot\|_{H_1\otimes H_2}$; $\|\cdot\|_{\mathcal{L}(H_1,H_2)}$ denotes the operator norm in the Banach space $\mathcal{L}(H_1, H_2)$ of all bounded and linear operators from H_1 into H_2.
Let K be a real separable Hilbert space. We recall that if A and B are linear bounded operators from K into K and B is Hilbert-Schmidt, then AB is also Hilbert-Schmidt and

$$\|AB\|_{K\otimes K} \leq \|A\|_{\mathcal{L}(K,K)} \|B\|_{K\otimes K}. \tag{1.4}$$

If L is a Hilbert-Schmidt operator from K into K, the *Carleman-Fredholm determinant* of $I + L$ is

$$\det{}_2(I + L) = \prod_{k\geq 1} (1 + \lambda_k) e^{-\lambda_k},$$

where λ_k are the eigenvalues of L, counted with respect to their multiplicity (see [26, Appendix A.2] and [10]).

We set $H = L^2(0,1)$ and consider also $H_0 = \{f \in \Omega :$ there exists the distributional derivative $f' \in H\}$. It is well known that any $f \in H_0$ is absolutely continuous and so differentiable a.e., with the derivative defined a.e. which coincides with the distributional derivative.

The space H_0 will be *considered isomorphic* to H and so identified (when no confusion may arise) with H through the isomorphism $f \mapsto f'$ from H_0 onto H; its inverse

mapping will be simply denoted by $\sim$, i.e., $\tilde{f}_t = (\tilde{f})_t = \int_0^t f_s ds, \quad f \in H, \quad t \in [0,1]$. By defining the inner product

$$\langle h, g \rangle_{H_0} := \langle h', g' \rangle_H, \quad f, g \in H_0,$$

H_0 becomes a real separable Hilbert space.

2 Preliminary results

In this section we introduce the basic boundary value problem studied in later sections, and give two equivalent integral formulations of it.

Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be a given continuous function.
An Borel set $\Omega_0 \subset \Omega$ is called *admissible* if $\mathbb{P}(\Omega_0) = 1$ and, moreover, for any $\omega \in \Omega_0$, $\mathbb{P}$-a.s., for any $h \in H_0$, we have that $\omega + h \in \Omega_0$ (i.e., $\omega + H_0 \subset \Omega_0$, for any $\omega \in \Omega_0$, $\mathbb{P}$-a.s.).
A *Borel measurable* mapping $X : \Omega \to \Omega$, $X = (X_t)$, $t \in [0,1]$, is said to be a *solution* of (1.1) if there exists an *admissible open set* $\Gamma \subset \Omega$, such that $X(\omega) \in C_0^1$, for any $\omega \in \Gamma$, and, for any $t \in [0,1]$, we have

$$X_t'(\omega) + \int_0^t f(s, X_s(\omega), X_s'(\omega))ds = X_0'(\omega) + \omega_t; \;\; X_0(\omega) = X_1(\omega) = 0, \;\; \omega \in \Gamma.$$

We say that *pathwise uniqueness* holds for (1.1) if given two solutions X and Z, we have $X = Z$, $\mathbb{P}$-a.s.; we say that *uniqueness in law* holds for (1.1) if given two solutions X and Z, they have the same law, i.e., for any $A \in \mathcal{F}$, we have $\mathbb{P}(X \in A) := \mathbb{P}(X^{-1}(A)) = \mathbb{P}(Z \in A)$.
In the sequel we will often omit dependence on ω of X and write, more shortly,

$$X_t' + \int_0^t f(s, X_s, X_s')ds = X_0' + \omega_t, \qquad X_0 = X_1 = 0, \;\; \omega \in \Gamma, \;\; t \in [0,1]. \tag{2.1}$$

Remark 2.1. Pathwise uniqueness is investigated by [16] always assuming $\Gamma = \Omega$; our generality is also motivated by the existence and uniqueness result in Section 4.6.

An easy equivalence between the classical and weak formulation of solutions is proved in the next result.

Proposition 2.2. *A Borel measurable mapping $X : \Omega \to \Omega$ such that $X(\omega) \in C_0^1$, for any $\omega \in \Gamma$ (Γ is an admissible open set in Ω), is a solution if and only if it satisfies, for every $\varphi \in C_0^1$,*

$$-\int_0^1 \varphi_t' X_t' dt + \int_0^1 \varphi_t f(t, X_t, X_t')dt + \int_0^1 \varphi_t' \omega_t dt = 0, \;\; \omega \in \Gamma. \tag{2.2}$$

Proof. We have to show equivalence between (2.1) and (2.2). It is clear that if X is a solution according to (2.1) then (multiplying by $\varphi \in C_0^1$ and integrating by parts) X is also a solution to (2.2).
Let now X be a solution according to (2.2). Letting

$$u_t = X_t' + \int_0^t f(s, X_s, X_s')ds - X_0' - \omega_t, \qquad t \in [0,1],$$

we obtain $\int_0^1 u_t \psi_t dt = 0$, for every $\psi \in C([0,1])$ with zero mean. This means that

$$\int_0^1 u_t \sin(2\pi n t) dt = \int_0^1 u_t \cos(2\pi n t) dt = 0, \qquad n \geq 1.$$

By the L^2-theory of Fourier series, u is a.e. constant; but since $u_0 = 0$ and u is continuous it must be $u_t = 0$ for every $t \in [0,1]$; it follows that X_t is a solution of (2.1). Alternatively, to prove that u is constant, one can use [4, Lemma VIII.1]. □

Following [16], we consider the solution Y to (1.1) corresponding to $f = 0$, i.e.,

$$Y_t(\omega) = -t \int_0^1 \omega_s ds + \int_0^t \omega_s ds, \qquad t \in [0,1],\ \omega \in \Omega. \tag{2.3}$$

Note that $Y : \Omega \to \Omega$ is a linear continuous and one to one mapping; moreover $Y(\Omega) = C_0^1$. Moreover, if $Y(\omega) = \eta$ then $Y_t^{-1}(\eta) = \omega_t = \eta_t' - \eta_0'$, $t \in [0,1]$.
Proposition 2.2 allows us to rewrite the boundary value problem (1.1) as an integral equation. Consider in fact the Green's function of $-d^2/dt^2$ (with Dirichlet boundary condition)

$$K(t,s) = t \wedge s - ts. \tag{2.4}$$

First note that

$$Y_t(\omega) = \int_0^1 \frac{\partial K}{\partial s}(t,s)\omega_s ds, \ \ t \in [0,1]. \tag{2.5}$$

Equivalently, using the stochastic Itô integral, we have, $\mathbb{P}$-a.s., $Y_t = -\int_0^1 K(t,s) d\omega_s$, $t \in [0,1]$.
Introducing the operator

$$\mathcal{K} : \Omega \to \Omega \qquad v \mapsto \int_0^1 K(\cdot, s) v_s ds, \tag{2.6}$$

we have the following standard result, whose proof is omitted for brevity (see also [7]).

Lemma 2.3. *A measurable mapping $X : \Omega \to \Omega$, such that $X(\omega) \in C_0^1$, for any $\omega \in \Gamma$ (Γ is an admissible open set in Ω) is a solution of* (1.1) *if and only if it solves the integral equation*

$$X(\omega) - \mathcal{K}(f(\cdot, X(\omega), X'(\omega))) = Y(\omega), \ \ \omega \in \Gamma. \tag{2.7}$$

Lemma 2.3 shows that the existence of solution to (1.1) is equivalent to the existence of a fixed point for the operator $X \mapsto \mathcal{K}(f(\cdot, X, X')) + Y(\omega)$, for any $\omega \in \Gamma$; such fixed point must also depend measurably on ω. By the properties of the Green's function, if $X = X(\omega)$ is a fixed point of this operator then necessarily $X_0 = 0 = X_1$.

As in [16] let us introduce the operator $T : \Omega \to \Omega$,

$$T_t(\omega) = \omega_t + \int_0^t f(s, Y_s(\omega), Y_s'(\omega)) ds, \ \ \omega \in \Omega,\ t \in [0,1]. \tag{2.8}$$

Note that T is continuous on Ω.
The following useful result is an extension of [16, Proposition 1.1]. It characterizes pathwise uniqueness for (1.1) by means of the mapping T. We provide a proof for the sake of completeness.

Proposition 2.4. *The following assertions are equivalent.*

(i) There exists an admissible open set $\Gamma \subset \Omega$ such that the mapping $T : T^{-1}(\Gamma) \to \Gamma$ is bijective.

(ii) There exists an admissible open set $\Gamma \subset \Omega$, such that, for any $\omega \in \Gamma$, there exists a unique function $u \in C_0^1$ which is a solution of

$$\begin{cases} u'_t + \int_0^t f(s, u_s, u'_s)ds = u'_0 + \omega_t \\ u_0 = 0 = u_1. \end{cases} \tag{2.9}$$

Moreover, if (i) (or (ii)) holds, then there exists a pathwise unique solution X to (1.1) *which is given by $X(\omega) = Y(T^{-1}(\omega))$, $\omega \in \Gamma$ and $X(\omega) = 0$ if $\omega \in \Omega \setminus \Gamma$.*

Proof. $(i) \Longrightarrow (ii)$. We first show the existence of a solution u corresponding to $\omega \in \Gamma$. Let $\eta = T^{-1}(\omega)$ and define $u := Y(T^{-1}(\omega))$. We find, for $t \in [0, 1]$,

$$u'_t = Y'_t(\eta) = -\int_0^1 \eta_s ds + \eta_t = Y'_0(\eta) + \omega_t - \int_0^t f(s, Y_s(\eta), Y'_s(\eta))ds$$

$$= u'_0 + \omega_t - \int_0^t f(s, u_s, u'_s)ds.$$

Uniqueness is obtained from the injectivity of T, using the following fact: if $u \in C_0^1$ is any solution to (2.9) with $\omega \in \Gamma$, then we have $T(Y^{-1}(u)) = \omega$ (see the comment after (2.3)).

$(ii) \Longrightarrow (i)$. Let us check that T is onto. For a fixed $\omega \in \Gamma$, let u be the solution corresponding to ω. We define $\eta_t = Y_t^{-1}(u) = u'_t - u'_0$, $t \in [0, 1]$. We immediately find $T(\eta) = \omega$. Let us verify that T is one to one. If $\eta = T(\omega_1) = T(\omega_2)$, then we have, for $k = 1, 2$,

$$\eta_t = \omega_k(t) + \int_0^t f(s, Y_s(\omega_k), Y'_s(\omega_k))ds, \quad t \in [0, 1].$$

Since $\omega_k(t) = Y'_t(\omega_k) - Y'_0(\omega_k)$, $t \in [0, 1]$, $k = 1, 2$, we see that $u^{(1)} = Y(\omega_1)$ and $u^{(2)} = Y(\omega_2)$ are two solutions to (2.9) (when $\omega = \eta$). It follows that $Y(\omega_1) = Y(\omega_2)$ and so $\omega_1 = \omega_2$.

To prove the final assertion, i.e., that the given X is in fact a solution, it remains to check that $X : \Gamma \to \Omega$ is Borel measurable. Since Y is continuous, the assertion holds if $T^{-1} : \Gamma \to T^{-1}(\Gamma)$ is measurable. To show this fact it is enough to apply an important theorem due to Kuratowski (see [19][Section 1.3]). This result states that any Borel measurable mapping φ from a complete separable metric space F_1 into another complete separable metric space F_2, which is also bijective from a Borel subset $E_1 \subset F_1$ onto a Borel subset $E_2 \subset F_2$, has the inverse $\varphi^{-1} : E_2 \to E_1$ which is Borel measurable (i.e., φ is a measurable isomorphism). $\square$

3 Pathwise Uniqueness

In this section we adapt techniques from the classical theory of boundary value problems to the integro-differential equation (2.1) and obtain sufficient conditions on the function f which guarantee the existence and pathwise uniqueness of the solution for *any* given $\omega \in \Omega$ (i.e., we can take, as it is done in [16], $\Gamma = \Omega$ in the definition of solution to (1.1)).

3.1 Existence and uniqueness under non-resonance conditions

Consider the boundary value problem

$$X'_t + \int_0^t f(s, X_s)ds = X'_0 + \omega_t, \qquad X_0 = X_1 = 0, \ \ \omega \in \Omega, \tag{3.1}$$

and assume that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and differentiable with respect to its second argument with bounded derivative.
By Lemma 2.3 and Proposition 2.4, solvability of (3.1) is proved if, for any $\omega \in \Omega$, there exists a unique function $u \in C^1_0$ which satisfies

$$u_t - \mathcal{K}(f(\cdot, u))(t) = \int_0^1 \frac{\partial K}{\partial s}(t,s)\omega_s ds, \ \ t \in [0,1]. \tag{3.2}$$

Write $H = L^2(0,1)$ and introduce

$$\Phi : H \longrightarrow H, \qquad u \mapsto f(\cdot, u(\cdot)). \tag{3.3}$$

Notice that the existence and uniqueness of the solution of (3.2) for every $\omega \in \Omega$ is guaranteed, *in particular,* if the map

$$(I - \mathcal{K}\Phi) : H \longrightarrow H, \qquad u \mapsto u - \mathcal{K}(f(\cdot, u(\cdot)))$$

is a global homeomorphism. In order to apply a variant of the abstract global implicit function theorem (cf. [6, Theorem 3.9, page 29]) to (3.3), we shall need the following

Lemma 3.1. *([6, Lemma 3.4, page 95]) Let M be a real Hilbert space and $K : M \to M$ be a compact, symmetric, positive definite operator. Let $0 < \lambda_1 \le \lambda_2 \le \dots \le \lambda_n \le \dots$ be its eigenvalues (counted according to their multiplicity). Consider a family $\mathcal{A}$ of symmetric linear operators on M, and assume that there exist μ_n, μ_{n+1}, such that*

$$\lambda_n I < \mu_n I \le A \le \mu_{n+1} I < \lambda_{n+1} I, \quad n \ge 1, \tag{3.4}$$

for each $A \in \mathcal{A}$. Then, the linear map $F : M \to M$, $x \mapsto x - KAx$, for each $A \in \mathcal{A}$ has a bounded inverse and there exists $N > 0$ such that

$$\|(I - KA)^{-1}\|_{\mathcal{L}(M,M)} \le N, \qquad \text{for all } A \in \mathcal{A}. \tag{3.5}$$

We can now state and prove the main result of this section.

Theorem 3.2. *Assume that*

$$\pi^2 m^2 < h \le \frac{\partial f}{\partial x}(t,x) \le k < \pi^2(m+1)^2, \quad t \in [0,1], \ x \in \mathbb{R}, \tag{3.6}$$

where $m \ge 0$ is an integer and h, k are real constants. Then (3.1) has a unique solution.

The assumption on $\frac{\partial f}{\partial x}$ is a non-resonance condition in the sense that zero is the only solution to the BVP associated to the linear problem $v''_t + \frac{\partial f}{\partial x}(\tau, \xi)v_t = 0$, for any fixed $\tau, \xi \in \mathbb{R}$.

Proof. We only give a sketch of the proof, since it is similar to the second proof of [6, Theorem 3.3, page 93]. This proof consists of an application of [6, Theorem 3.9, page 29] and Lemma 3.1. As mentioned above, we have to show that $(I - \mathcal{K}\Phi)$ is a global homeomorphism from H onto H. To this end, it is sufficient to check that Φ in (3.3) is of class C^1 on H and that $(I - \mathcal{K}D\Phi(u))^{-1}$ exists, for any $u \in H$ ($D\Phi(u)$ being the Fréchet derivative of Φ at $u \in H$) and satisfies, for some $N > 0$, the inequality

$$\|(I - \mathcal{K}D\Phi(u))^{-1}\|_{\mathcal{L}(H,H)} \leq N, \qquad \text{for all } u \in H. \tag{3.7}$$

From the assumptions on f, it follows that Φ is of class C^1. In order to verify (3.7), it suffices to apply Lemma 3.1 with $M = H, K = \mathcal{K}$, $\lambda_n = (n\pi)^2$, taking as $\mathcal{A}$ the family of all bounded linear operators on H defined by $Ay(t) = D\Phi(u)[y](t) = \frac{\partial f}{\partial x}(t, u(t))y(t)$, for every $u \in H$. It is clear that the non-resonance hypothesis allows us to apply Lemma 3.1. □

We close this section with a short discussion of the Fredholm alternative in our context. Consider a linear BVP for which

$$f(t, X_t, X'_t) = \mu X_t, \tag{3.8}$$

with $\mu > 0$ a real positive constant. By Lemma 2.3 we know that (1.1) with (3.8) is equivalent to the linear integral equation

$$(I - \mu\mathcal{K})X(\omega) = Y(\omega), \quad \omega \in \Omega, \tag{3.9}$$

with Y given by (2.5). The operator $\mathcal{K}$ is self-adjoint in $L^2(0,1)$ and the eigenvalues and eigenfunctions of $\mathcal{K}$ are $\frac{1}{\mu} = \frac{1}{k^2\pi^2}$ and $\sin(k\pi t)$, with k integer, $k \geq 1$. The classical Fredholm alternative states that, if $\mu \neq k^2\pi^2$, then $\ker(I - \mu\mathcal{K}) = \{0\}$ and (2.1) admits a unique solution $X(\omega) = (I - \mu\mathcal{K})^{-1}Y(\omega)$, while for $\mu = k^2\pi^2$ there exist solutions if and only if Y is orthogonal in L^2 to the eigenfunctions of $\mathcal{K}$.
In our case, the requirement that Y be orthogonal in $L^2(0,1)$ to the eigenfunctions of $\mathcal{K}$ yields

$$\begin{aligned} \int_0^1 Y_t(\omega)\sin(k\pi t)\,dt &= -\int_0^1 \sin(k\pi t)\Big(\int_0^1 K(t,s)d\omega_s\Big)dt \\ = -\int_0^1 \Big(\int_0^1 K(t,s)\sin(k\pi t)dt\Big)d\omega_s &= -\frac{1}{k^2\pi^2}\int_0^1 \sin(k\pi s)d\omega_s = 0. \end{aligned} \tag{3.10}$$

However, the stochastic integral $\frac{1}{k^2\pi^2}\int_0^1 \sin(k\pi s)\,d\omega_s$ is a non-degenerate gaussian random variable (with mean 0 and variance $\frac{1}{k^4\pi^4}\int_0^1 \sin^2(k\pi s)\,ds = \frac{1}{2k^4\pi^4}$). It follows that the probability that (3.10) is verified vanishes. This implies that (3.8) does not have a solution, for $\sqrt{\mu} = k\pi$.
Hence, we have proved

Proposition 3.3. *(i) If $\mu \neq n^2\pi^2$, the linear Dirichlet BVP associated to (3.8) has a unique solution.*
(ii) If $\mu = m^2\pi^2$ for some $m \geq 1$, the linear Dirichlet BVP associated to (3.8) has no solution.

Remark 3.4. As in the deterministic case, the above result can be also deduced from the explicit expression of the solution using Fourier series.

Remark 3.5. A standard argument shows that the above result still holds in the general case

$$f(t, X_t, X'_t) = aX_t + bX'_t, \qquad a, b \in \mathbb{R}, \tag{3.11}$$

where the condition for the existence and uniqueness of the solution of (3.1) is now $a - b^2/4 \neq k^2\pi^2$, with $k \in \mathbb{Z}$.

3.2 Existence and uniqueness under Lipschitz-type conditions

In this section we give some other existence and pathwise uniqueness results for our BVP, taking into account Proposition 2.4 and using some tools of the theory of classical nonlinear ODEs. To this end, we will consider the solution Y (see (2.3)). Let $\omega \in \Omega$ and define $\hat{f} : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ by

$$\hat{f}(t, x, y) := f(t, x + Y_t(\omega), y + Y'_t(\omega)).$$

A straightforward computation leads to

Lemma 3.6. *Let $\omega \in \Omega$ be fixed. A function $u \in C_0^1$ is a solution of*

$$\begin{cases} u'_t + \int_0^t f(s, u_s, u'_s) ds = u'_0 + \omega_t \\ u_0 = 0 = u_1 \end{cases} \tag{3.12}$$

if and only if $z_t := u_t - Y_t(\omega)$ belongs to $C^2([0,1])$ and is a solution of

$$\begin{cases} z''_t + \hat{f}(t, z_t, z'_t) = 0 \\ z_0 = 0 = z_1. \end{cases} \tag{3.13}$$

Note that, as a consequence of its definition, the function $\hat{f}$ has the same regularity of f with respect to the second and third arguments.

Lemma 3.6 allows to apply the classical existence and uniqueness results for boundary value problems by Bailey, Shampine and Waltman [3]. To do this, let K, L be real numbers and define

$$\alpha(L, K) = \begin{cases} \frac{2}{\sqrt{4K - L^2}} \arccos \frac{L}{2\sqrt{K}} & \text{if } 4K - L^2 > 0 \\ \frac{2}{\sqrt{L^2 - 4K}} \operatorname{arccosh} \frac{L}{2\sqrt{K}} & \text{if } 4K - L^2 < 0, L > 0, K > 0 \\ \frac{2}{L} & \text{if } 4K - L^2 = 0, L > 0 \\ +\infty & \text{otherwise} \end{cases} \tag{3.14}$$

and

$$\beta(L, K) = \alpha(-L, K). \tag{3.15}$$

The first result of [3] that we use here is based on the contraction mapping principle, and its proof consists in showing the existence and uniqueness of a fixed point of an operator defined through the Green's function for problem (3.13) (analogue to the integral operator introduced in Section 2). However, more work is needed in order to get an optimal result.

Theorem 3.7. *([3, Theorem 3.5]). Assume that there exist K, L such that*

$$|\hat{f}(t, x, y) - \hat{f}(t, \tilde{x}, \tilde{y})| \leq K|x - \tilde{x}| + L|y - \tilde{y}|, \tag{3.16}$$

for all $t \in [0,1]$ and for all $x, \tilde{x}, y, \tilde{y} \in \mathbb{R}$. Assume also that $1 < 2\alpha(K, L)$. Then (3.13) has a unique solution.

Remark 3.8. The above result is optimal, in the sense that neither existence nor uniqueness are guaranteed when $1 = 2\alpha(K, L)$.

Recalling Proposition 2.4 and Lemma 3.6, we obtain

Corollary 3.9. *Assume that there exist K, L such that*

$$|f(t, x, y) - f(t, \tilde{x}, \tilde{y})| \leq K|x - \tilde{x}| + L|y - \tilde{y}|, \tag{3.17}$$

for all $t \in [0, 1]$ and for all $x, \tilde{x}, y, \tilde{y} \in \mathbb{R}$. Assume also that $1 < 2\alpha(K, L)$. Then (1.1) *has a unique solution. In particular, if*

$$|f(t, x, y) - f(t, \tilde{x}, \tilde{y})| \leq L(|x - \tilde{x}| + |y - \tilde{y}|), \tag{3.18}$$

for all $t, x, \tilde{x}, y, \tilde{y}$ and $0 < L < 4$, then (1.1) *has a unique solution.*

Proof. It is sufficient to apply Theorem 3.7, Lemma 3.6 and the definition of $\hat{f}$. As for the particular case when (3.18) holds, it is easy to check that if $0 < L < 4$ then we can get

$$1 < \frac{4}{\sqrt{4L - L^2}} \arccos \frac{\sqrt{L}}{2}. \tag{3.19}$$

From the definition of α it follows that the above inequality is equivalent to $1 < 2\alpha(L, L)$ and thus Theorem 3.7 applies with $K = L$. $\square$

Corollary 3.9 improves Proposition 1.4 in [16], which shows existence and uniqueness under the assumption that

$$|f(t, x, y) - f(t, \tilde{x}, \tilde{y})| \leq L(|x - \tilde{x}| + |y - \tilde{y}|), \tag{3.20}$$

for all $t \in [0, 1]$ and for all $x, \tilde{x}, y, \tilde{y} \in \mathbb{R}$, and $L < 1/3$.
Corollary 3.9 can be further improved by means of a generalized Lipschitz condition. To this end, we recall

Theorem 3.10. *([3, Theorem 7.6]). Assume that $\hat{f}$ is locally Lipschitz and that there exist K, L_1, L_2 such that*

$$\hat{f}(t, x, y) - \hat{f}(t, \tilde{x}, y) \leq K(x - \tilde{x}), \tag{3.21}$$

for all $x \geq \tilde{x}, t \in [0, 1], y \in \mathbb{R}$,

$$L_1(y - \tilde{y}) \leq \hat{f}(t, x, y) - \hat{f}(t, x, \tilde{y}) \leq L_2(y - \tilde{y}), \tag{3.22}$$

for all $y \geq \tilde{y}, t \in [0, 1], x \in \mathbb{R}$. Assume also that $1 < \alpha(L_2, K) + \beta(L_1, K)$. Then (3.13) has a unique solution.

Arguing as above, we obtain

Corollary 3.11. *Assume that f is locally Lipschitz and that there exist K, L_1, L_2 such that*

$$f(t, x, y) - f(t, \tilde{x}, y) \leq K(x - \tilde{x}), \tag{3.23}$$

for all $x \geq \tilde{x}, t \in [0, 1], y \in \mathbb{R}$,

$$L_1(y - \tilde{y}) \leq f(t, x, y) - f(t, x, \tilde{y}) \leq L_2(y - \tilde{y}), \tag{3.24}$$

for all $y \geq \tilde{y}, t \in [0, 1], x \in \mathbb{R}$. Assume also that $1 < \alpha(L_2, K) + \beta(L_1, K)$. Then (3.12) has a unique solution.

Corollary 3.11 can be compared with Proposition 1.3 in [16], where it is assumed that $f = f(x, y)$ is nonincreasing in each coordinate and that it has linear growth. More precisely, the monotonicity condition in x, y is contained in (3.23), (3.24) when we take $K = 0$ and $L_2 = 0$, respectively. Moreover, it follows from the definitions that $\beta(L_1, 0) = +\infty$. Notice that no linear growth restriction is required in Corollary 3.11; the assumptions are satisfied also (as remarked in [3]) by a nonlinearity of the form $f(t, x) = -e^x$.

4 Uniqueness in law

In this section we will give sufficient conditions to have uniqueness in law for solutions to the BVP associated to equation (1.1). These conditions are not covered by the pathwise uniqueness results of previous sections. In this section (excluding Remark 4.24) we will always assume that

Hypothesis 4.1. *The function $f : [0,1] \times \mathbb{R}^2$ is continuous and bounded and has first and second spatial partial derivatives f_x, f_y, f_{xx}, f_{xy} and f_{yy} which are continuous and bounded.*

4.1 $H-$differentiability

Let $H = L^2(0,1)$ and H_0 be the subspace of Ω introduced at the end of Section 1. Recall that a Hilbert-Schmidt operator $K : H \to H$ can be represented by a Kernel $K(t, s) \in L^2[(0,1)^2]$, i.e., $K_t h = \int_0^1 K(t, s) h_s ds$, $t \in [0,1]$. Identifying H with H_0, a Hilbert-Schmidt operator $R : H_0 \to H_0$ can be represented by a Kernel $R(t, s) \in L^2[(0,1)^2]$ as follows:

$$R_t f = \int_0^t dr \int_0^1 R(r, s) f'_s ds, \quad f \in H_0, \; t \in [0,1]. \tag{4.1}$$

In the sequel we will identify Hilbert-Schmidt operators from H_0 into H_0 with their corresponding kernels in $L^2[(0,1)^2]$; to stress this fact, we will also write $H_0 \otimes H_0 \simeq L^2[(0,1)^2]$. The following definition is inspired from [22] (compare also with [14, Chapter 4], [26, Section 3.3] and [26, Definition B.6.2]).

Definition 4.2. *Let K be a real separable Hilbert space. A measurable map $\mathcal{G} : \Omega \to K$ is said to be H-differentiable if the following conditions hold:*

(1) For any $\omega \in \Omega$, $\mathbb{P}$-a.s., the mapping $\mathcal{G}(\omega + \cdot) : H_0 \to K$, $h \mapsto \mathcal{G}(\omega + h)$, is Fréchet differentiable on H_0.

(2) For any $\omega \in \Omega$, $\mathbb{P}$-a.s., the H-derivative $D_H\mathcal{G}(\omega)$, which is defined by

$$D_H\mathcal{G}(\omega)[h] = \lim_{r \to 0} \frac{\mathcal{G}(\omega + rh) - \mathcal{G}(\omega)}{r}, \quad h \in H_0, \tag{4.2}$$

is a Hilbert-Schmidt operator from H_0 into K.

(3) the map $\omega \mapsto D_H\mathcal{G}(\omega)$ is measurable from Ω into $H_0 \otimes K$.

Remark 4.3. In condition (1) we are requiring that $\mathcal{G}$ is differentiable along the directions of H_0 (the Cameron-Martin space or the space of admissible shifts for $\mathbb{P}$, see [26]). The space H_0 is densely and continuously embedded in Ω (the immersion $i : H_0 \to \Omega$ is even compact). The triple $(\Omega, H_0, \mathbb{P})$ is an important example of abstract Wiener space (see [14, Section 4.1]). The notion of H-differentiability can be more generally formulated in abstract Wiener spaces.

In the special case when $K = H_0$ we obtain (see (4.1) and compare with [16, Theorem 2.1])

Definition 4.4. *A measurable map $\mathcal{G} : \Omega \to H_0$ is said to be H-differentiable if the following conditions hold:*

(1) For any $\omega \in \Omega$, $\mathbb{P}$-a.s., the mapping $\mathcal{G}(\omega + \cdot) : H_0 \to H_0$ is Fréchet differentiable on H_0.

(2) For any $\omega \in \Omega$, $\mathbb{P}$-a.s., there exists the H-derivative, i.e., a kernel $D_H\mathcal{G}(\omega) \in L^2([0,1]^2)$, such that, for any $\omega \in \Omega$, $\mathbb{P}$-a.s.,

$$\lim_{r \to 0} \frac{\mathcal{G}(\omega + rh) - \mathcal{G}(\omega)}{r} = \int_0^{\cdot} (D_H\mathcal{G}(\omega))[h'](s)ds, \quad h \in H_0, \tag{4.3}$$

where $(D_H\mathcal{G}(\omega))[h'](t) = \int_0^1 D_H\mathcal{G}(\omega)(t,s)h'_s ds$, $t \in [0,1]$.

(3) the map $\omega \mapsto D_H\mathcal{G}(\omega)$ is measurable from Ω into $L^2([0,1]^2)$.

The concept of H-differentiability goes back to Gross at the beginning of the 60s and it is now well understood that it is strictly related to Malliavin Calculus (see also Appendix A). The relation between the H-differentiability and Malliavin derivative is completely clarified in [22] (see also [14, Section 4.1.3]). It turns out that $D_H\mathcal{G}$ is *the Malliavin derivative of $\mathcal{G}$*. More precisely, we have the following result as a special case of [22, Theorem 3.1].

Theorem 4.5. *(Sugita [22]) Let K be a real separable Hilbert space. Let us consider a measurable map $\mathcal{G} : \Omega \to K$ which is H-differentiable and such that $\mathcal{G} \in L^2(\Omega; K)$ and*

$$D_H\mathcal{G} \in L^2(\Omega; H_0 \otimes K).$$

Then $\mathcal{G}$ belongs to $D^{1,2}(K)$ (see Appendix A). Moreover, we have $D_M\mathcal{G} = D_H\mathcal{G}$, $\mathbb{P}$-a.s..

Let us go back to the map T given in (2.8); $T : \Omega \to \Omega$, $T = I + G$, where $G : \Omega \to H_0$,

$$G_t(\omega) = \int_0^t f(s, Y_s(\omega), Y'_s(\omega))ds, \quad \omega \in \Omega, \ \ t \in [0,1]. \tag{4.4}$$

We have the following lemma.

Lemma 4.6. *The following assertions hold:*

(i) The mapping $T : \Omega \to \Omega$ is continuously Fréchet differentiable on Ω, with Fréchet derivative $DT(\omega) : \Omega \to \Omega$,

$$\begin{aligned} DT(\omega)[\theta] &= \theta + \int_0^{\cdot} \Big(f_x(s, Y_s(\omega), Y'_s(\omega))\, Y_s(\theta) + f_y(s, Y_s(\omega), Y'_s(\omega)) Y'_s(\theta) \Big) ds \\ &= \theta + DG(\omega)[\theta], \ \ \omega, \, \theta \in \Omega. \end{aligned}$$

(ii) The mapping $G : \Omega \to H_0$ is H-differentiable, with the following H-derivative $D_HG(\omega)$, for any $\omega \in \Omega$,

$$\begin{aligned} D_HG(\omega)[h](t) &= f_x(t, Y_t(\omega), Y'_t(\omega))\, Y_t(\tilde{h}) + f_y(t, Y_t(\omega), Y'_t(\omega))\, Y'_t(\tilde{h}) \\ &= -a_t(\omega) \int_0^1 K(t,s)h_s ds - b_t(\omega) \int_0^1 \partial_t K(t,s) h_s ds, \ \ h \in H, \ t \in [0,1], \end{aligned}$$

where $a_t = a_t(\omega) = f_x(t, Y_t(\omega), Y'_t(\omega))$ and $b_t = b_t(\omega) = f_y(t, Y_t(\omega), Y'_t(\omega))$. Moreover, the following relation between Fréchet and H-derivative holds:

$$DG(\omega)[h](t) = \int_0^t D_HG(\omega)[h'](s)ds, \quad h \in H_0, \ \ t \in [0,1], \ \ \omega \in \Omega. \tag{4.5}$$

Proof. (i) It is straightforward to check that T is continuously Fréchet differentiable on Ω. First one verifies its Gâteaux-differentiability at a fixed ω, finding the Gâteaux derivative $DT(\omega)$. The computations are easy, we only note the estimate

$$\sup_{s,r\in[0,1]} |Y_s(\omega + r\theta)| \le \|\omega\|_\infty + \|\theta\|_\infty.$$

Then one proves in a straightforward way that the mapping: $\omega \mapsto DT(\omega)$ from Ω into $\mathcal{L}(\Omega)$ ($\mathcal{L}(\Omega)$ denotes the Banach space of all linear and bounded operators from Ω into Ω endowed with the operator norm) is continuous and this gives the assertion.
(ii) First note that the operator

$$h \mapsto D_H G(\omega)[h] = -a_t(\omega)\int_0^1 K(t,s)h_s ds - b_t(\omega)\int_0^1 \partial_t K(t,s)h_s ds, \quad h \in H,$$

is a Hilbert-Schmidt operator on H. To check the H-differentiability of G, it is enough to verify that (the limit is in H)

$$\lim_{r\to 0} \frac{G'_t\left(\omega + r\int_0^\cdot h_s ds\right) - G'_t(\omega)}{r} = f_x(t, Y_t(\omega), Y'_t(\omega))\, Y_t(\tilde h) + f_y(t, Y_t(\omega), Y'_t(\omega))\, Y'_t(\tilde h), \tag{4.6}$$

$h \in H$, where $\tilde h_t = \int_0^t h_s ds$, and also that

$$h \mapsto D_H G\left(\omega + \int_0^\cdot h_s ds\right) \quad \text{is continuous from } H \text{ into } L^2([0,1]^2), \tag{4.7}$$

for any $\omega \in \Omega$. The proof of (4.6) is straightforward (formula (4.6) also appears in [16]) and also the verification of (4.7).

It remains to show the measurability property, i.e., that $\omega \mapsto D_H G(\omega)$ is measurable from Ω into $L^2([0,1]^2)$. We fix an orthonormal basis (e_i) in H and consider the orthonormal basis $(e_i \otimes e_j)$ in $L^2([0,1]^2)$; recall that $e_i \otimes e_j(t,s) = e_i(t)e_j(s)$, $s,t \in [0,1]$ (cf. see [21, Chapter VI]). To obtain the measurability property, it is enough to verify that, for any $i,j \ge 1$, the mapping:

$$\omega \mapsto \int_0^1\int_0^1 DF(\omega)(s,t)e_i(t)e_j(s)dtds \tag{4.8}$$

is measurable from Ω into $\mathbb{R}$ and this follows easily. The proof is complete. $\square$

Remark 4.7. *We have, for any $\omega \in \Omega$,*

$$\|D_H G(\omega)\|_{L^2([0,1]^2)} \le (\|f_x\|_0 + \|f_y\|_0)\,(\|K\|_{L^2([0,1]^2)} + \|\partial_t K\|_{L^2([0,1]^2)}). \tag{4.9}$$

Lemma 4.8. *For any $\omega \in \Omega$, the Fréchet derivative $DT(\omega) : \Omega \to \Omega$ is such that*

$$\begin{aligned} DT(\omega) = I + DG(\omega) : \Omega \to \Omega \ \textit{is an isomorphism} \ \Leftrightarrow \\ \textit{the linearized equation } u''_t + b_t u'_t + a_t u_t = 0,\ \ u_0 = u_1 = 0, \\ \textit{with } a_t = a_t(\omega) = f_x(t, Y_t(\omega), Y'_t(\omega)),\ b_t = b_t(\omega) = f_y(t, Y_t(\omega), Y'_t(\omega)), \\ \textit{has the unique zero solution.} \end{aligned} \tag{4.10}$$

Proof. Since $DG(\omega)$ is a compact operator on Ω, by the Fredholm alternative theorem it is enough to check that $I + DG(\omega)$ is one to one. Fix ω and let $\theta \in \Omega$ be such that

$$\theta_t + \int_0^t \big(f_x(s, Y_s(\omega), Y_s'(\omega))\, Y_s(\theta) + f_y(s, Y_s(\omega), Y_s'(\omega))\, Y_s'(\theta)\big) ds = 0, \quad t \in [0,1].$$

It follows that θ is differentiable and

$$\theta_t' + a_t(\omega)\, Y_t(\theta) + b_t(\omega)\, Y_t'(\theta) = 0.$$

Recalling that $\theta_t' = Y_t''(\theta)$, we find that $Y_t(\theta) = u_t$ solves the boundary value problem $u_t'' + a_t u_t + b_t u_t' = 0$, $u_0 = u_1 = 0$. Hence $Y(\theta) = 0$ and so $\theta = 0$. $\square$

4.2 An anticipative Girsanov theorem involving a Carleman-Fredholm determinant

Here we present a non-adapted version of the Girsanov theorem proved recently in [25, Theorem 3.3]. This result will be used in the sequel to prove uniqueness in law for our boundary value problem (1.1). Its formulation requires some concepts of Malliavin Calculus (see Appendix A). Recall that $H_0 \otimes H_0 \simeq L^2[(0,1)^2]$.

Hypothesis 4.9.

(i) Let $F : \Omega \to H_0$ be a measurable mapping which belongs to $D^{2,2}(H_0)$.

(ii) If $\delta(F)$ denotes the Skorohod integral of F and $D_M F$ its Malliavin derivative, it holds

$$\exp\Big(- \delta(F) + \|D_M F\|_{L^2([0,1]^2)}\Big) \in L^4(\Omega). \tag{4.11}$$

Let us comment the previous assumptions; (i) and (ii) are immediately obtained from the corresponding assumptions in [25, Theorem 3.2] with $r = 2$ and $\gamma = 3$. Consider $\Lambda_F : \Omega \to \mathbb{R}$,

$$\Lambda_F(\omega) = \det{}_2(I + D_M F(\omega))\, \exp\Big(- \delta(F)(\omega) \; - \; \frac{1}{2}|F(\omega)|_{H_0}^2\Big). \tag{4.12}$$

As pointed out after [25, Theorem 3.2] (see also Appendix A.2 in [26]) under Hypothesis 4.9 we have Λ_F, $\Lambda_F(I + D_M F)^{-1} v \in L^4(\Omega)$, for any $v \in H_0$.

Theorem 4.10. *(Üstünel-Zakai [25])*

(H1) Assume that F satisfies Hypothesis 4.9 and consider the associated measurable transformation $\mathcal{T} = \mathcal{T}_F : \Omega \to \Omega$,

$$\mathcal{T}(\omega) = \omega + F(\omega), \quad \omega \in \Omega. \tag{4.13}$$

(H2) Assume that, for any $\omega \in \Omega$, $\mathbb{P}$-a.s., $[I + D_M F(\omega)] : H_0 \to H_0$ is an isomorphism (here $I = I_{H_0}$).

(H3) Assume that there exists a measurable (left inverse) transformation $\mathcal{T}_l : \Omega \to \Omega$ such that

$$\mathcal{T}_l(\mathcal{T}(\omega)) = \omega, \quad \omega \in \Omega, \ \mathbb{P} - a.s..$$

Then there exists a (Borel) probability measure $\mathbb{Q}$ on Ω, which is equivalent to the Wiener measure $\mathbb{P}$, having density $\frac{d\mathbb{Q}}{d\mathbb{P}} = \Lambda_F$, and such that

$$\mathbb{Q}(\mathcal{T}^{-1}(A)) = \mathbb{Q}(\{\omega \in \Omega \, : \, \mathcal{T}(\omega) \in A\}) = \mathbb{P}(A), \ \textit{for any Borel set } A \subset \Omega. \tag{4.14}$$

Note that the assertion says that the process $(\mathcal{T}_t(\omega))_{t\in[0,1]}$ is a Wiener process on $(\Omega, \mathcal{F}, \mathbb{Q})$. The measure $\mathbb{Q}$ is called a Girsanov measure in [25].

Remark 4.11. It is useful to compare the previous theorem with another non-adapted extension of the Girsanov theorem known as the Ramer-Kusuoka theorem (see [12], [14, Theorem 4.1.2] and [26, Section 3.5]). This result has been also applied in [2], [7], [8] and [16]. Its formulation requires the following assumptions.

(H1) Assume that $F : \Omega \to H_0$ is H-differentiable and that the mapping: $h \mapsto D_H F(\omega + h)$ is continuous from H_0 into $H_0 \otimes H_0$, for any $\omega \in \Omega$, $\mathbb{P}$-a.s..

(H2) Assume that the measurable transformation $\mathcal{T} = I + F : \Omega \to \Omega$ (see (4.13)*) is* bijective.

(H3) Assume that, for any $\omega \in \Omega$, $\mathbb{P}$-a.s., $[I + D_H F(\omega)] : H_0 \to H_0$ is an isomorphism.

If (H1)-(H3) hold, then there exists a (Borel) probability measure $\mathbb{Q}$ on Ω, which is equivalent to $\mathbb{P}$, having density $\frac{d\mathbb{Q}}{d\mathbb{P}} = |\Lambda_F|$, such that (4.14) *holds.*

Note that Theorem 4.10 does not require the invertibility of $\mathcal{T}$. On the other hand, additional integrability assumptions on F are imposed. There is also a difference in the expression of $\frac{d\mathbb{Q}}{d\mathbb{P}}$. Indeed Theorem 4.10 claims that $\det_2(I + D_H F)$ is positive, $\mathbb{P}$-a.s., while in the Ramer-Kusuoka theorem, we have to consider $|\det_2(I + D_H F)|$.

4.3 Some results on H-differentiability and Malliavin derivatives

Let $X = (X_t)$, $X : \Omega \to \Omega$ be a measurable transformation. We introduce an associated measurable mapping $S^X = S : \Omega \to \Omega$, as follows

$$\begin{aligned} S_t(\omega) &= \omega_t - \int_0^t f(s, X_s(\omega), X'_s(\omega))ds = [(I+F)(\omega)]_t, \quad \text{where } F = F^X : \Omega \to H_0, \\ F_t(\omega) &= -\int_0^t f(s, X_s(\omega), X'_s(\omega))ds, \quad t \in [0,1]. \end{aligned} \tag{4.15}$$

Proposition 4.12. *A measurable mapping $X : \Omega \to \Omega$ is a solution if and only if there exists an admissible open set $\Gamma \subset \Omega$, such that*

$$X_t(\omega) = Y_t(S(\omega)), \quad \omega \in \Gamma, \quad t \in [0,1].$$

Proof. Recall that $Y_t(\omega) = -t\int_0^1 \omega_s ds + \int_0^t \omega_s ds$, so that

$$Y_t(\omega) = \int_0^1 \partial_s\big(t \wedge s - ts\big)\,\omega_s ds.$$

Let X be a solution. By Lemma 2.3 we have, for any $\omega \in \Gamma$,

$$X_t(\omega) = \int_0^1 \partial_s\big(t \wedge s - ts\big)\left(\omega_s - \int_0^s f(r, X_r(\omega), X'_r(\omega))dr\right) ds = Y_t(S(\omega)).$$

The reverse implication follows similarly. $\square$

Let us go back to the continuous map $T : \Omega \to \Omega$. Recall that pathwise uniqueness can be characterized by the fact that T is *bijective* (see the precise statement in Proposition 2.4). In this section we are mainly interested in situations in which we do not know if T is bijective or not.

The following two results will be important. The first one says that T is always a measurable *left inverse of S* (compare with Theorem 4.10).

Lemma 4.13. *Let X be a solution to* (2.1) *and let S be the associated measurable mapping (see* (4.15)*). We have on the admissible open set $\Gamma \subset \Omega$ (see* (2.1)*)*

$$T \circ S = I; \tag{4.16}$$

in particular S is always injective on Γ and T surjective from $S(\Gamma)$ onto Γ.

Proof. We have, for any $\omega \in \Gamma$, using Proposition 4.12,

$$T_t(S(\omega)) = S_t(\omega) + \int_0^t f(s, Y_s(S(\omega)), Y_s'(S(\omega)))ds$$
$$= \omega_t - \int_0^t f(s, Y_s(S(\omega)), Y_s'(S(\omega)))ds + \int_0^t f(s, X_s(\omega), X_s'(\omega))ds = \omega_t, \quad t \in [0,1].$$

□

We introduce now an assumption on solutions to the boundary value problem under consideration. Let X be a solution to (2.1). We say that X *satisfies the hypothesis (L)* if there exists an admissible Borel set $\Omega_0 \subset \Omega$ such that

$$\textbf{(L)} \begin{cases} \text{for any } \omega \in \Omega_0, \text{ the linearized BVP } u_t'' + b_t u_t' + a_t u_t = 0, \;\; u_0 = u_1 = 0, \\ \text{where } a_t = a_t(\omega) = f_x(t, X_t(\omega), X_t'(\omega)) \;\text{ and } b_t = b_t(\omega) = f_y(t, X_t(\omega), X_t'(\omega)) \\ \textit{has only the zero solution.} \end{cases} \tag{4.17}$$

If $T : \Omega \to \Omega$ is bijective (as it is always the case in [16]) a condition which implies (L) is

$$\textbf{(LY)} \begin{cases} \text{for any } \omega \in \Omega, \text{ the linearized BVP } u_t'' + b_t u_t' + a_t u_t = 0, \;\; u_0 = u_1 = 0, \\ \text{where } a_t = a_t(\omega) = f_x(t, Y_t(\omega), Y_t'(\omega)) \;\text{ and } b_t = b_t(\omega) = f_y(t, Y_t(\omega), Y_t'(\omega)) \\ \textit{has only the zero solution.} \end{cases} \tag{4.18}$$

Using Lemmas 4.6 and 4.8 we can prove the following result (recall the admissible open set $\Gamma \subset \Omega$ given in (2.1) and the fact that $T = I + G$ in (4.4)).

Theorem 4.14. *Assume Hypothesis 4.1. Let X be a solution to* (2.1) *which satisfies (L) and let $S = I + F$ be the associated measurable mapping (see (4.15)). Then the map F is H-differentiable and we have, for any $\omega \in \Omega$, $\mathbb{P}$-a.s.,*

$$[D_H F(\omega)] = [I + D_H G(S(\omega))]^{-1} - I = -D_H G(S(\omega)) \left(I + D_H G(S(\omega))\right)^{-1}. \tag{4.19}$$

Moreover, for any $\omega \in \Omega$, $\mathbb{P}$-a.s. (setting $I = I_{H_0}$),

$$[I + D_H F(\omega)] : H_0 \to H_0 \quad \textit{is an isomorphism.}$$

Proof. The proof is divided into some steps.
I Step. We show that there exists an admissible open set $\Gamma_0 \subset \Gamma$, such that S and F are Fréchet differentiable at any $\omega \in \Gamma_0$.

According to formula (4.10) the Fréchet derivative $DT(S(\omega))$ is an isomorphism from Ω into Ω if and only if (4.17) holds for ω (recall that $X = Y \circ S$). Let $\Omega_0 \subset \Omega$ be the

admissible Borel set such that (4.17) holds for any $\omega \in \Omega_0$. Define $\Omega' = \Omega_0 \cap \Gamma$. Clearly $\mathbb{P}(\Omega') = 1$ and also $H_0 + \omega' \subset \Omega'$, for any $\omega \in \Omega'$, $\mathbb{P}$-a.s.. Thus Ω' is an admissible Borel set in Ω.

Fix $\omega \in \Omega'$. Since $DT(S(\omega))$ is an isomorphism, we can apply the inverse function theorem and deduce that T is a local diffeomorphism from an open neighborhood $U_{S(\omega)}$ of $S(\omega)$ into an open neighborhood $V_{T(S(\omega))} = V_\omega$ of $T(S(\omega)) = \omega$. We may also assume that $V_\omega \subset \Gamma$, for any $\omega \in \Omega'$. Let us denote by T^{-1} the local inverse function (we have $T^{-1}(V_\omega) = U_{S(\omega)}$). By Proposition 4.12, we know that

$$\{\theta \in \Gamma \,:\, S(\theta) \in T^{-1}(V_\omega)\} = V_\omega.$$

It follows that S is Fréchet differentiable in any $\omega' \in V_\omega$ and that

$$DS(\omega') = (DT(S(\omega')))^{-1} = (I + DG(S(\omega')))^{-1}.$$

Introduce the open set

$$\Gamma_0 = \bigcup_{\omega \in \Omega'} V_\omega \subset \Gamma.$$

Since $\Omega' \subset \Gamma_0$, we have that $\mathbb{P}(\Gamma_0) = 1$. In addition $H_0 + \omega \subset \Gamma_0$, for any $\omega \in \Gamma_0$, $\mathbb{P}$-a.s.. The restriction of S to Γ_0 is a Fréchet-differentiable function with values in Ω. It follows that also F is Fréchet differentiable at any $\omega \in \Gamma_0$ with Fréchet derivative

$$DF(\omega) = (I + DG(S(\omega)))^{-1} - I. \tag{4.20}$$

II Step. We check that, for any $\omega \in \Gamma_0$, $DF(\omega)[h] \in H_0$, if $h \in H_0$, and, moreover, for any $\omega \in \Gamma_0$, $DF(\omega) \in H_0 \otimes H_0$ (when considered as an operator from H_0 into H_0). We also show that, for any $\omega \in \Gamma_0$, $\mathbb{P}$-a.s., the map:

$$DF(\omega + \cdot) : H_0 \to H_0 \otimes H_0 \ \text{ is continuous} \tag{4.21}$$

and that $DF(\cdot)$ is measurable from Γ_0 into $H_0 \otimes H_0$.

Let us consider, for $\omega \in \Gamma_0$, $k = (I + DG(S(\omega)))^{-1}[h]$. We have $k + DG(S(\omega))[k] = h$. It follows that $k \in H_0$, since $DG(S(\omega))[k] \in H_0$. By (4.5) in Lemma 4.6, we obtain that if $h \in H_0$, then

$$(I + DG(S(\omega)))^{-1}[h] = (I + D_H G(S(\omega)))^{-1}[h].$$

By using the identity

$$(I + D_H G(S(\omega)))^{-1} - I = -D_H G(S(\omega))(I + D_H G(S(\omega)))^{-1}, \ \ \omega \in \Gamma_0,$$

since $(I + D_H G(S(\omega)))^{-1}$ is a bounded operator and $D_H G(S(\omega))$ is Hilbert-Schmidt, we deduce that $(I + D_H G(S(\omega)))^{-1} - I$ is a Hilbert-Schmidt operator on H_0 (see (1.4)). We verify now the continuity property (4.21), i.e., that for any $\omega \in \Gamma_0$, $\mathbb{P}$-a.s., for any $k \in H_0$,

$$\lim_{h \to k,\, h \in H_0} D_H G(S(\omega+h))(I + D_H G(S(\omega+h)))^{-1} = D_H G(S(\omega+k))(I + D_H G(S(\omega+k)))^{-1}$$

(note that, for any $\omega \in \Gamma_0$, $\mathbb{P}$-a.s., $DF(\omega + h)$ is well-defined at any $h \in H_0$). This requires the following considerations.

(a) The mapping: $D_H G : \Omega \to H_0 \otimes H_0$ is continuous. Indeed we know (see Lemma 4.6)

$$D_H G(\omega) = -f_x(t, Y_t(\omega), Y'_t(\omega))\, K(t,s) - f_y(t, Y_t(\omega), Y'_t(\omega))\, \partial_t K(t,s)$$

(identifying operators in $H_0 \otimes H_0$ with corresponding kernels in $L^2([0,1]^2)$). Since Y and Y' are continuous from Ω into Ω we get easily our assertion using Hypothesis 4.1.

(b) Since $S : \Gamma_0 \to \Omega$ is continuous and Γ_0 is admissible, we get that, for any $\omega \in \Gamma_0$, $\mathbb{P}$-a.s., the map: $S(\omega + \cdot) : H_0 \to \Omega$ is continuous. Using also (a), we obtain that, for any $\omega \in \Gamma_0$, $\mathbb{P}$-a.s., $(D_H G \circ S)(\omega + \cdot) : H_0 \to H_0 \otimes H_0$ is continuous.

(c) To get the assertion we use (1.4) and the following fact: for any $\omega \in \Gamma_0$, $\mathbb{P}$-a.s., we have

$$\lim_{h \to k} (I + D_H G(S(\omega + h)))^{-1} = (I + D_H G(S(\omega + k)))^{-1}$$

(limit in $\mathcal{L}(H_0, H_0)$) for any $k \in H_0$. This holds since, for any $\omega \in \Gamma_0$, $\mathbb{P}$-a.s., $(I + D_H G(S(\omega + h)))$ is invertible for any $h \in H_0$, and, moreover, for any $\omega \in \Gamma_0$, $\mathbb{P}$-a.s., $\lim_{h \to k}(I + D_H G(S(\omega + h))) = (I + D_H G(S(\omega + k)))$ in $\mathcal{L}(H_0, H_0)$, for any $k \in H_0$.

To check the measurability property, we can repeat the argument before formula (4.8).

III Step. There exists $c_0 > 0$, depending on $\|f_x\|_0$ and $\|f_y\|_0$ such that, for any $\omega \in \Gamma_0$,

$$|DS(\omega)h|_{H_0} = |(I + D_H G(S(\omega)))^{-1} h|_{H_0} \le c_0 |h|_{H_0}, \quad h \in H_0. \tag{4.22}$$

This estimate follows from Corollary 5.2 applied to $L = D_H G(S(\omega))$.

IV Step. We prove that F is H-differentiable with $D_H F(\omega) = DF(\omega)$ (see (4.1)), for any $\omega \in \Gamma_0$.

The assertion will be proved if we show that there exists, for any $\omega \in \Gamma_0$, $R(\omega) \in H_0 \otimes H_0$, such that

$$\lim_{r \to 0} \frac{F(\omega + rh) - F(\omega)}{r} = R(\omega)[h], \quad h \in H_0 \tag{4.23}$$

(the limit is in H_0). Indeed, once this is checked we will get that $R(\omega) = DF(\omega)$ (because the topology of H_0 is stronger than the one in Ω). Moreover, we will obtain (since Γ_0 is admissible) that, for any $\omega \in \Gamma_0$, $\mathbb{P}$-a.s., $F(\omega + \cdot) : H_0 \to H_0$ is Gâteaux differentiable on H_0. Combining this fact with (4.21), we will deduce the required property (1) in Definition 4.4.

To prove (4.23), we first show that, for any $t \in [0,1]$, $\omega \in \Gamma_0$, and $h \in H_0$,

$$(i)\ \lim_{r \to 0} \frac{X_t(\omega + rh) - X_t(\omega)}{r} = Y_t(DS(\omega)[h]), \tag{4.24}$$

$$(ii)\ \lim_{r \to 0} \frac{X'_t(\omega + rh) - X'_t(\omega)}{r} = Y'_t(DS(\omega)[h]).$$

Let us only check (ii) (the proof of (i) is similar). Using the fact that $X = Y \circ S$ on Γ_0, we have (for r small enough)

$$\frac{X'_t(\omega + rh) - X'_t(\omega)}{r} = -\int_0^1 \Big(\frac{S_s(\omega + rh) - S_s(\omega)}{r}\Big) ds + \frac{S_t(\omega + rh) - S_t(\omega)}{r}$$

and the assertion follows passing to the limit as $r \to 0$ (using also (4.22)).

Let us go back to (4.23). Define, for $\omega \in \Gamma_0$, and $h \in H_0$,

$$R(\omega)[h](t) = \int_0^t \Big(a_s(\omega) Y_s(DS(\omega)[h]) + b_s(\omega) Y_s'(DS(\omega)[h]) \Big) ds, \quad t \in [0,1].$$

We have

$$\lim_{r\to 0} \Big| \frac{F(\omega + rh) - F(\omega)}{r} - R(\omega)[h] \Big|_{H_0}^2$$

$$= \lim_{r \to 0} \int_0^1 \Big| - \frac{f(s, X_s(\omega + rh), X_s'(\omega + rh)) - f(s, X_s(\omega), X_s'(\omega))}{r}$$

$$-a_s(\omega) Y_s(DS(\omega)[h]) - b_s(\omega) Y_s'(DS(\omega)[h]) \Big|^2 ds.$$

Now an application of the dominated convergence theorem shows that the previous limit exists and is 0. The proof is complete. □

Next we provide useful properties of the Malliavin derivative of F, taking advantage of the techniques in [10] (see Appendix B). The first one is an L^∞-estimate for $D_H F$ and will be important in Section 4.5.

Proposition 4.15. *Under the assumptions of Theorem 4.14, there exists $C > 0$, depending on $\|f_x\|_0$ and $\|f_y\|_0$, such that, for any $\omega \in \Omega$, $\mathbb{P}$-a.s. (identifying $L^2([0,1]^2)$ with $H_0 \otimes H_0$),*

$$\|D_H F(\omega)\|_{L^2([0,1]^2)} \le C. \tag{4.25}$$

Proof. Using (1.4), estimates (4.9) and (4.22) lead to the assertion. □

The following result provides an "explicit expression" for the Malliavin derivative $D_H F$. The formula follows from (4.19) and Theorem 5.3.

Proposition 4.16. *Under the assumptions of Theorem 4.14 (identifying $H_0 \otimes H_0$ with $L^2([0,1]^2)$), we have, for any $y \in L^2(0,1)$, $\omega \in \Omega$, $\mathbb{P}$-a.s.,*

$$D_H F(\omega)[y] = - \int_0^1 \gamma(t,s) y(s) ds, \quad t \in [0,1],$$

with

$$\gamma(t,s) = \begin{cases} (\frac{1}{W})[a_t u_2(s) \psi(t) + b_t u_2(s) \psi'(t)], & 0 \le s < t \le 1, \\ (\frac{1}{W})(a_t u_2(t) + b_t u_2'(t)) \varphi(s), & 0 \le t < s \le 1. \end{cases}$$

Here $u_k, k = 1, 2$, denote the solutions to $u_k'' + b_t u_k' + a_t u_k = 0$ (the coefficients a_t and b_t depend on ω and are given in (4.17)) with initial conditions $u_1(0) = u_2'(0) = 1$, $u_1'(0) = u_2(0) = 0$, respectively. Moreover, $W = u_1 u_2' - u_2 u_1'$, $M = u_1(1)/u_2(1)$, and

$$\varphi(s) = -u_2(s) M + u_1(s), \ \psi(t) = u_2(t) M - u_1(t), \ \ t \in [0,1], \ \ s \in [0,1].$$

The next result is needed in Section 4.5.

Proposition 4.17. *Under the assumptions of Theorem 4.14, we have that $F \in D^{2,2}(H_0)$.*

Proof. The proof is divided into some steps.

I Step. We check that $G \in D^{2,2}(H_0)$.

Since we already now that $G \in D^{1,2}(H_0)$, we only need to show that $D_H G \in D^{1,2}(H_0 \otimes H_0)$. Applying Theorem 4.5, it is enough to prove that $D_H G : \Omega \to H_0 \otimes H_0$ is H-differentiable and that $D_H(D_H G) \in L^\infty(\Omega, \mathcal{HS}(H_0, H_0 \otimes H_0))$. We proceed similarly to the proof of Lemma 4.6 (with more involved computations). Recall that $H_0 \otimes H_0 \simeq L^2([0,1]^2)$. First we introduce a suitable operator $R(\omega) \in \mathcal{HS}(H_0, H_0 \otimes H_0)$, for any $\omega \in \Omega$. This operator can be identified with an integral operator acting from $L^2(0,1)$ into $L^2([0,1]^2)$, i.e., with a kernel in $L^2([0,1]^3)$. For any $\omega \in \Omega$, we set

$$c_t = c_t(\omega) = f_{xx}(t, Y_t(\omega), Y_t'(\omega)), \;\; d_t = d_t(\omega) = f_{xy}(t, Y_t(\omega), Y_t'(\omega)),$$

$$e_t = e_t(\omega) = f_{yy}(t, Y_t(\omega), Y_t'(\omega)).$$

Now $R(\omega)$ can be identified with the following kernel in $L^2([0,1]^3)$:

$$c_t K(t,s)K(t,r) + d_t\, \partial_t K(t,s)\, K(t,r) + d_t K(t,s)\, \partial_t K(t,r) + e_t\, \partial_t K(t,s)\, \partial_t K(t,r),$$

$t, s, r \in [0,1]$. We have, for any $h \in H$, $\omega \in \Omega$,

$$\lim_{r \to 0} \Big| \frac{D_H G\left(\omega + r \int_0^{\cdot} h_s ds\right) - D_H G(\omega)}{r} - R(\omega)[h] \Big|_{L^2([0,1]^2)} = 0, \;\; h \in H.$$

It is easy to check that $h \mapsto \; R\left(\omega + \int_0^{\cdot} h_s ds\right)$ is continuous from H into $L^2([0,1]^3)$, for any $\omega \in \Omega$. In addition the mapping $\omega \to R(\omega)$ is measurable from Ω into $L^2([0,1]^3)$ (this can be done using the argument before formula (4.8)). This shows that $D_H G$ is H-differentiable and moreover that $D_H^2 G(\omega) = R(\omega)$, $\omega \in \Omega$. Finally, it is easy to see that $D_H^2 G \in L^\infty(\Omega, L^2([0,1]^3))$ (recall that $L^2([0,1]^3) \simeq \mathcal{HS}(H_0, H_0 \otimes H_0)$).

II Step. We prove that $D_H F$ is H-differentiable.

In order to check condition (1) in Definition 4.2, we use the admissible open set $\Gamma_0 \subset \Omega$ given in the proof of Theorem 4.14 and prove that, for any $\omega \in \Gamma_0$, $\mathbb{P}$-a.s., the mapping:

$$h \mapsto D_H F(\omega + h)$$

from H_0 into $H_0 \otimes H_0$ is Fréchet differentiable on H_0.

Let us consider a Borel set $\Omega'' \subset \Gamma_0$, with $\mathbb{P}(\Omega'') = 1$ such that, for any $\omega \in \Omega''$, $\omega + H_0 \subset \Gamma_0$. Fix any $\omega \in \Omega''$. We would like to differentiate in formula (4.19), i.e., to differentiate the mapping

$$h \mapsto (I + D_H G(S(\omega + h)))^{-1} - I \tag{4.26}$$

from H_0 into $H_0 \otimes H_0$, applying the usual composition rules for Fréchet derivatives. The only problem is that the mapping $h \mapsto S(\omega + h) = \omega + h + F(\omega + h)$ does not take values in H_0. This is the reason for which we will verify directly the Fréchet differentiability at a fixed $h_0 \in H_0$. By setting $(I + D_H G(S(\omega + h))) = M(h)$, we have, for any $h \in H_0$,

$$M^{-1}(h) - M^{-1}(h_0) = M^{-1}(h)\big(M(h_0) - M(h)\big)M^{-1}(h_0)$$

$$= -M^{-1}(h)\big(D_H G([S(\omega + h) - S(\omega + h_0)] + S(\omega + h_0)) - D_H G(S(\omega + h_0))\big)M^{-1}(h_0)$$

$$= -M^{-1}(h)\Big(D_H^2 G(S(\omega + h_0))\big[S(\omega + h) - S(\omega + h_0)\big]\Big)M^{-1}(h_0)$$

$$
\begin{aligned}
&\quad + M^{-1}(h)\, o([S(\omega+h) - S(\omega+h_0)])\, M^{-1}(h_0) \\
&= -M^{-1}(h)\Big(D_H^2 G(S(\omega+h_0))\big\{(h-h_0) + D_H F(\omega+h_0)[h-h_0]\big\}\Big)M^{-1}(h_0) \\
&\quad - M^{-1}(h)\Big(D_H^2 G(S(\omega+h_0))\,[o(h-h_0)]\Big)M^{-1}(h_0) \\
&\quad + M^{-1}(h)\, o([S(\omega+h) - S(\omega+h_0)])M^{-1}(h_0),
\end{aligned}
$$

as $h \to h_0$; we have used I Step together with the fact that $S(\omega+h) - S(\omega+h_0) = (h-h_0) + \big(F(\omega+h) - F(\omega+h_0)\big) \in H_0$ and $S(\omega+h) - S(\omega+h_0) = (h-h_0) + D_H F(\omega+h_0)[h-h_0] + o(h-h_0)$ as $h \to h_0$. This shows the Fréchet differentiability of the mapping in (4.26) at h_0, with Fréchet derivative along the direction $k \in H_0$ given by

$$
V(\omega)[k] = -M^{-1}(h_0)\,\Big((D_H^2 G(S(\omega+h_0))\big[k + D_H F(\omega+h_0)[k]\big]\Big)\,M^{-1}(h_0).
$$

Let (e_j) be an orthonormal basis in H_0. Using (1.4), we find, for any $j \geq 1$,

$$
\begin{aligned}
\|V(\omega)[e_j]\|_{H_0\otimes H_0} &\leq \|M^{-1}(h_0)\|^2_{\mathcal{L}(H_0,H_0)}\big(\|(D_H^2 G(S(\omega+h_0))[e_j]\|_{H_0\otimes H_0} \qquad (4.27)\\
&\quad + \|D_H^2 G(S(\omega+h_0))\|_{\mathcal{L}(H_0,H_0\otimes H_0)}\, |D_H F(\omega+h_0)[e_j]|_{H_0}\big).
\end{aligned}
$$

It follows that, for any $\omega \in \Omega''$, $V(\omega) \in \mathcal{HS}(H_0, H_0 \otimes H_0)$. Up to now we know that condition (1) in Definition 4.2 holds for $\mathcal{G} = D_H F$, with $D_H(D_H F)(\omega) = V(\omega)$, $\omega \in \Omega''$. It remains to check that $V(\cdot)$ is measurable from Ω'' into $\mathcal{HS}(H_0, H_0 \otimes H_0)$. This holds if, for any $k \in H_0$, the mapping:

$$
\omega \mapsto V(\omega)[k]
$$

is measurable from Ω'' into $\mathcal{HS}(H_0, H_0)$ and this is easy to check. The assertion is proved.

III Step. We prove that $D_H(D_H F) \in L^\infty(\Omega, \mathcal{HS}(H_0, H_0 \otimes H_0))$.

By Theorem 4.5 this will imply that $F \in D^{2,2}(H_0)$. Taking into account the bounds (4.22) and (4.25) and the fact that $D_H^2 G \in L^\infty(\Omega, \mathcal{HS}(H_0, H_0 \otimes H_0)))$, we find (see (4.27)), for any $\omega \in \Omega$, $\mathbb{P}$-a.s.,

$$
\|V(\omega)\|^2_{\mathcal{HS}(H_0,H_0\otimes H_0)} = \sum_{j\geq 1} \|V(\omega)[e_j]\|^2_{H_0\otimes H_0} \;\leq\; C,
$$

where $C > 0$ depends on $\|f_x\|_0$, $\|f_y\|_0$, $\|f_{xx}\|_0$, $\|f_{xy}\|_0$ and $\|f_{yy}\|_0$. The proof is complete. □

4.4 Exponential integrability of the Skorohod integral $\delta(F)$

We start with a technical result from [14, Section 3.1] which requires to introduce the space $L^{1,2}$ (see [14, page 42]).

A real stochastic process $u \in L^2([0,1] \times \Omega)$ belongs to the class $L^{1,2}$ if, for almost all $t \in [0,1]$, $u_t \in D^{1,2}(\mathbb{R})$, and there exists a measurable version of the two-parameter process $D_M u_t$ which still belongs to $L^2([0,1] \times \Omega)$. One can prove that $L^{1,2} \subset Dom(\delta)$. Moreover $L^{1,2}$ is a Hilbert space and has norm

$$
\|u\|^2_{L^{1,2}} = \|u\|^2_{L^2([0,1]\times\Omega)} + \|D_M u\|^2_{L^2([0,1]\times\Omega)}.
$$

Let $u \in L^{1,2}$. Fix a partition π of $[0,1]$, $\pi = \{t_0 = 0 < t_1 < \ldots < t_N = 1\}$. Let $|\pi| = \sup_{0 \le i \le N-1} |t_{i+1} - t_i|$ and define the following random variable

$$\hat{S}^{\pi}(\omega) = \sum_{i=0}^{N-1} \frac{1}{t_{i+1} - t_i} \Big(\int_{t_i}^{t_{i+1}} \mathbb{E}\big[u_s / \mathcal{F}_{[t_i, t_{i+1}]^c}\big](\omega)\, ds \Big) \left(\omega(t_{i+1}) - \omega(t_i)\right), \;\; \omega \in \Omega,$$

$\mathbb{P}$-a.s.; here $\mathbb{E}\big[u_s / \mathcal{F}_{[t_i,t_{i+1}]^c}\big]$ denotes the conditional expectation of $u_s \in L^2(\Omega)$ with respect to the σ-algebra $\mathcal{F}_{[t_i,t_{i+1}]^c}$ (where $[t_i, t_{i+1}]^c = [0,1] \setminus [t_i, t_{i+1}]$). This is the σ-algebra (completed with respect to $\mathbb{P}$) generated by the random variables $\int_0^1 1_A(s)\, d\omega_s$, when A varies over all Borel subsets of $[t_i, t_{i+1}]^c$ (see [14, page 33]).

According to [14, page 173], when $u \in L^{1,2}$ there exists a sequence of partitions (π^n) such that $\lim_{n \to \infty} |\pi^n| = 0$ and

$$\hat{S}^{\pi^n} \to \delta(u), \;\; \text{as } n \to \infty, \;\; \mathbb{P} - a.s. \;\text{ and in }\;\; L^2(\Omega). \tag{4.28}$$

We can now prove the following estimate.

Proposition 4.18. *Let $u \in L^{1,2} \cap L^{\infty}([0,1] \times \Omega)$. Then, for any $a > 0$, we have*

$$\mathbb{E}[\exp(a\, |\delta(u)|\,)] \le 2 e^{\frac{a^2\, \|u\|_\infty^2}{2}},$$

where $\|u\|_\infty = \|u\|_{L^\infty([0,1]\times\Omega)}$.

Proof. We will use assertion (4.28), with the previous notation. It is enough to prove the following bound, for any $n \ge 1$,

$$\mathbb{E}[\exp(a\, |\hat{S}^{\pi^n}|\,)] \le 2 e^{a^2\, \frac{\|u\|_\infty^2}{2}}. \tag{4.29}$$

Once (4.29) is proved, an application of the Fatou lemma will allow us to get the assertion.

By elementary properties of conditional expectation, we have, for almost all $s \in [0,1]$, ω, $\mathbb{P}$-a.s.,

$$|\mathbb{E}\big[u_s / \mathcal{F}_{[t_0,t_1]^c}\big]| \le \|u\|_\infty,$$

for any $0 \le t_0 < t_1 \le 1$. It follows that, for any $n \ge 1$, ω, $\mathbb{P}$-a.s.,

$$\Big| \frac{1}{t_{i+1}^n - t_i^n} \int_{t_i^n}^{t_{i+1}^n} \mathbb{E}\big[u_s / \mathcal{F}_{[t_i^n, t_{i+1}^n]^c}\big](\omega)\, ds \Big| \le \|u\|_\infty.$$

Setting $Z_{i,n} = \frac{1}{t_{i+1}^n - t_i^n} \int_{t_i^n}^{t_{i+1}^n} \mathbb{E}\big[u_s / \mathcal{F}_{[t_i^n, t_{i+1}^n]^c}\big](\omega)\, ds$, we get

$$\mathbb{E}[\exp(a\, |\hat{S}^{\pi^n}|\,)] = \mathbb{E}\Big[e^{a |\sum_{i=0}^{N_n} Z_{i,n} (\omega(t_{i+1}^n) - \omega(t_i^n))|} \Big] \le \mathbb{E}\Big[e^{a \sum_{i=0}^{N_n} |Z_{i,n}| |\omega(t_{i+1}^n) - \omega(t_i^n)|} \Big]$$

$$\le \mathbb{E}\Big[e^{a \|u\|_\infty \sum_{i=0}^{N_n} |\omega(t_{i+1}) - \omega(t_i)|} \Big] = \mathbb{E}\Big[\prod_{i=0}^{N_n} e^{a\|u\|_\infty |\omega(t_{i+1}^n) - \omega(t_i^n)|} \Big]$$

$$= \prod_{i=0}^{N_n} \mathbb{E}\Big[e^{a\|u\|_\infty |\omega(t_{i+1}^n) - \omega(t_i^n)|} \Big] = \prod_{i=0}^{N_n} \mathbb{E}\Big[e^{a\|u\|_\infty |\omega(t_{i+1}^n - t_i^n)|} \Big]$$

(in the last step we have used the independence of increments and stationarity of the Wiener process). Now the bound (4.29) follows easily, noting that

$$\mathbb{E}\Big[e^{c|\omega(t)|}\Big] \leq 2e^{\frac{c^2 t}{2}}, \quad c > 0, \ \ t \geq 0.$$

Indeed, we have, for any $n \geq 1$,

$$\mathbb{E}[\exp(a\,|\hat{S}^{\pi^n}|\,)] \leq \prod_{i=0}^{N_n} 2\mathbb{E}\Big[e^{\frac{a^2}{2}\|u\|_\infty^2\,(t_{i+1}^n - t_i^n)}\Big] = 2e^{a^2\,\frac{\|u\|_\infty^2}{2}}.$$

□

Identifying $F_t(\omega) = -\int_0^t f(s, X_s(\omega), X_s'(\omega))ds$, $t \in [0,1]$, with the associated stochastic process $u \in L^{1,2}$

$$u(t,\omega) = f(t, X_t(\omega), X_t'(\omega)), \ \ t \in [0,1], \ \ \omega \in \Omega$$

(see also [14, Section 4.1.4]) and applying the previous result, we obtain

Corollary 4.19. *Assume that $f : \mathbb{R} \to \mathbb{R}$ is a bounded function. Then, for any $a > 0$, it holds:*

$$\mathbb{E}[\exp(a\,|\delta(F)|\,)] \leq 2e^{\frac{a^2}{2}\|f\|_0^2}. \tag{4.30}$$

4.5 The main results

We state now our main result. This theorem implies as a corollary that uniqueness in law holds for our boundary value problem (1.1) in the class of solutions such that the corresponding linearized equations (see condition (L) in (4.17)) have only the zero solution. *Hence uniqueness in law holds for* (1.1) *whenever all solutions X to* (1.1) *satisfy (L).* For a concrete example, we refer to Section 4.6.
We remark that a statement similar to the result below is given in [16, Theorem 2.3] *assuming in addition that there is pathwise-uniqueness* for the boundary value problem (1.1). Indeed pathwise uniqueness and uniqueness for the linearized equation (see (4.18)) lead by the Ramer-Kusuoka theorem (see Remark 4.11) to Theorem 2.3 in [16]. More information on [16, Theorem 2.3] are collected in Remark 4.22.

Theorem 4.20. *Assume Hypothesis 4.1. Suppose that there exists a solution X to* (2.1) *such that (L) in* (4.17) *holds.*
Then there exists a probability measure $\tilde{\mathbb{Q}}$ on $(\Omega, \mathcal{F})$, which is equivalent to $\mathbb{P}$, having (positive $\mathbb{P}$-a.s.) density

$$\frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}} = \eta = \det{}_2(I + D_H G)\, \exp\Big(- \delta(G) \, - \, \frac{1}{2}|G|_{H_0}^2\Big) \tag{4.31}$$

(G is defined in (4.4)), such that the law of X under $\mathbb{P}$ is the same of Y under $\tilde{\mathbb{Q}}$, i.e.,

$$\mathbb{P}(\omega \, : \, X(\omega) \in A) = \mathbb{P}(X \in A) = \tilde{\mathbb{Q}}(Y \in A), \quad A \in \mathcal{F}. \tag{4.32}$$

Proof. Part I. We verify applicability of Theorem 4.10 with

$$\mathcal{T} := S = S^X$$

(S is defined in (4.15) and $S = I + F$). First we have that hypothesis (H3) of Theorem 4.10 holds with $\mathcal{T}_l = T$ by Lemma 4.13 (T is defined in (2.8)). Moreover, also (H2) holds by Theorem 4.14. It remains to check (H1), i.e., assumptions (i) and (ii) in Hypothesis 4.9. Note that (i) holds by Corollary 4.17. The main point is to check (ii). By (4.25), we easily find that

$$\exp\Big(\|D_M F\|_{L^2}^2\Big) \in L^4(\Omega).$$

Thus to prove (4.11) it remains to check that $\exp(-\delta(F)) \in L^4(\Omega)$ and this follows from Corollary 4.19.

Part II. We introduce the measure $\tilde{\mathbb{Q}}$ and establish (4.31) (without proving the positivity of η).

Recall that Theorem 4.10 says that

$$\mathbb{P}(A) = \mathbb{Q}(S^{-1}(A)), \quad A \in \mathcal{F}, \tag{4.33}$$

where $\mathbb{Q}$ is a probability measure on $(\Omega, \mathcal{F})$, equivalent to $\mathbb{P}$, with the following (positive $\mathbb{P}$-a.s.) density

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \Lambda_F = \det{}_2(I + D_M F)\, \exp\Big(-\delta(F) \, - \, \frac{1}{2}|F|_{H_0}^2\Big);$$

recall that $X = Y \circ S$, i.e., $X_t(\omega) = Y_t(S(\omega))$, $\omega \in \Omega$, $t \in [0,1]$, and so (see (4.4) and (4.15))

$$F = -G \circ S.$$

We denote by $\mathbb{E}^{\mathbb{P}}$ and $\mathbb{E}^{\mathbb{Q}}$ the expectations with respect to $\mathbb{P}$ and $\mathbb{Q}$.

Let $A \in \mathcal{F}$. Introducing $\Lambda_F^{-1} : \Omega \to \mathbb{R}_+$, where $\Lambda_F^{-1}(\omega) = \frac{1}{\Lambda_F(\omega)}$ if $\Lambda_F(\omega) > 0$ and 0 otherwise (see [26, Section 1.1]), we find

$$\begin{aligned}
\mathbb{P}(X \in A) &= \mathbb{P}(\omega : Y(S(\omega)) \in A) = \mathbb{P}(\omega : S(\omega) \in Y^{-1}(A)) \\
&= \mathbb{E}^{\mathbb{P}}[1_{(S(\omega) \in Y^{-1}(A))}] = \mathbb{E}^{\mathbb{P}}\Big[1_{(S(\omega) \in Y^{-1}(A))}\, \frac{d\mathbb{Q}}{d\mathbb{P}} \frac{d\mathbb{P}}{d\mathbb{Q}}\Big] \\
&= \mathbb{E}^{\mathbb{Q}}\Big[1_{(S(\omega) \in Y^{-1}(A))}\, \frac{d\mathbb{P}}{d\mathbb{Q}}\Big] = \mathbb{E}^{\mathbb{Q}}\Big[1_{(S(\omega) \in Y^{-1}(A))}\, \Lambda_F^{-1}\Big] \\
&= \mathbb{E}^{\mathbb{Q}}\Big[1_{(S(\omega) \in Y^{-1}(A))}\, (\det{}_2(I + D_H F))^{-1}\, \exp\Big(\delta(F) \, + \, \frac{1}{2}|F|_{H_0}^2\Big)\Big].
\end{aligned}$$

By the properties of the Carleman-Fredholm determinant (see [26, Lemma A.2.2]), setting $R = D_H F(\omega)$, $\omega \in \Omega$, we know that

$$(\det{}_2(I + R))^{-1} = \det{}_2\big((I + R)^{-1}\big) \exp\big(\mathrm{Trace}(R^2\, (I + R)^{-1})\big),$$

where $\mathrm{Trace}(R^2\, (I + R)^{-1})$ denotes the trace of the trace class (or nuclear) operator $R^2\, (I + R)^{-1}$ (recall that the composition of two Hilbert-Schmidt operators is a trace class operator). Using (4.19), and the fact that $\mathrm{Trace}(MN) = \mathrm{Trace}(NM)$, for any Hilbert-Schmidt operators M and N, we get

$$\begin{aligned}
\mathbb{P}(X \in A) &= \mathbb{E}^{\mathbb{Q}}\Big[1_{Y^{-1}(A)}(S(\cdot))\, \det{}_2(I + D_H G(S(\cdot))) \cdot \\
&\cdot \exp\Big(\mathrm{Trace}\big((D_H G)(S(\cdot))^2\, (I + D_H G(S(\cdot)))^{-1}\big)\Big) \cdot \exp\Big(-\delta(G \circ S) \, + \, \frac{1}{2}|G \circ S|_{H_0}^2\Big)\Big].
\end{aligned}$$

Now remark the law $\mathbb{P}_0$ of S under $\mathbb{P}$, i.e., $\mathbb{P}_0(A) = \mathbb{P}(S^{-1}(A))$, $A \in \mathcal{F}$, is equivalent to $\mathbb{P}$ by Theorem 4.10 [25, Lemma 2.1]. Using this fact we can apply Theorem B.6.4 in [26] and obtain the following identity ($\mathbb{P}$-a.s. and so also $\mathbb{Q}$-a.s.)

$$\delta(G \circ S) = (\delta(G)) \circ S - \langle G \circ S, F \rangle_{H_0} - \mathrm{Trace}\big((D_H G)(S(\cdot))\, D_H F\big).$$

$$= (\delta(G)) \circ S - \langle G \circ S, F \rangle_{H_0} + \mathrm{Trace}\big((D_H G)(S(\cdot))^2\, (I + D_H G(S(\cdot)))^{-1}\big).$$

We get, since $F = -G \circ S$,

$$\begin{aligned} \mathbb{P}(X \in A) &= \mathbb{E}^{\mathbb{Q}}\Big[1_{Y^{-1}(A)}(S(\cdot))\ \mathrm{det}_2(I + D_H G(S(\cdot))) \cdot \\ &\quad \cdot \exp\Big(- (\delta(G)) \circ S + \langle G \circ S, F \rangle_{H_0} \ + \frac{1}{2}|G \circ S|^2_{H_0}\Big)\Big] \\ &= \mathbb{E}^{\mathbb{Q}}\Big[1_{Y^{-1}(A)}(S(\cdot))\ \mathrm{det}_2(I + D_H G(S(\cdot))) \cdot \\ &\quad \cdot \exp\Big(- (\delta(G)) \circ S - \langle G \circ S, G \circ S \rangle_{H_0} + \frac{1}{2}|G \circ S|^2_{H_0}\Big) \\ &= \mathbb{E}^{\mathbb{Q}}\Big[1_{Y^{-1}(A)}(S(\cdot))\ \mathrm{det}_2(I + D_H G(S(\cdot)))\ \exp\Big(- (\delta(G)) \circ S \ - \ \frac{1}{2}|G \circ S|^2_{H_0}\Big). \end{aligned}$$

The previous calculations show that

$$\mathrm{det}_2(I + D_H G(S(\cdot)))\ \exp\Big(- (\delta(G)) \circ S \ - \ \frac{1}{2}|G \circ S|^2_{H_0}\Big) = \eta \circ S \in L^1(\Omega, \mathbb{Q})$$

and that it is positive $\mathbb{Q}$-a.s. (or $\mathbb{P}$-a.s.). Using that $\mathbb{Q}$ is a Girsanov measure (i.e., that the law of S under $\mathbb{Q}$ is $\mathbb{P}$), it is is elementary to check that $\eta \in L^1(\Omega, \mathbb{P})$ and moreover

$$\mathbb{P}(X \in A) = \mathbb{E}^{\mathbb{P}}\Big[1_A(Y)\ \mathrm{det}_2(I + D_H G)\ \exp\Big(- \delta(G) \ - \ \frac{1}{2}|G|^2_{H_0}\Big)\Big]. \tag{4.34}$$

Up to now we know that $\eta \in L^1(\Omega)$ and $\mathbb{E}^{\mathbb{P}}[\eta] = 1$.

Part III. It remains to show that $\eta > 0$, $\mathbb{P}$-a.s., i.e., that $\gamma = \mathrm{det}_2(I + D_H G) > 0$, $\mathbb{P}$-a.s. By Theorem 4.10, we know that $\mathrm{det}_2(I + D_H F) > 0$, $\mathbb{P}$-a.s. (or $\mathbb{Q}$-a.s.). This is equivalent to say that $\gamma \circ S > 0$, $\mathbb{P}$-a.s.. Assume by contradiction that there exists $A \in \mathcal{F}$ with $\mathbb{P}(A) > 0$ such that $\gamma(\omega) \leq 0$, for any $\omega \in A$. We have

$$0 \geq \mathbb{E}^{\mathbb{P}}[1_A \cdot \gamma] = \mathbb{E}^{\mathbb{Q}}[1_A(S(\cdot))\gamma(S(\cdot))].$$

But $\mathbb{E}^{\mathbb{Q}}[1_A(S(\cdot))\gamma(S(\cdot))]$ is positive if $\mathbb{Q}(S^{-1}(A)) > 0$. This holds, since $\mathbb{E}^{\mathbb{Q}}[1_{S^{-1}(A)}] = \mathbb{E}^{\mathbb{Q}}[1_A(S(\cdot))] = \mathbb{E}^{\mathbb{P}}[1_A] > 0$. We have found a contradiction. The proof is complete. □

The assertion of the theorem implies that $\mathrm{det}_2(I + D_H G) > 0$, $\mathbb{P}$-a.s.. This means that under the assumptions of Theorem 4.20 we have that condition (LY) in (4.18) holds $\mathbb{P}$-a.s..

Since η in Theorem 4.20 does not depend on X, we get immediately

Corollary 4.21. *Assume Hypothesis 4.1. Suppose that we have two solutions to* (1.1), *X^1 and X^2, which both satisfy hypothesis (L) in* (4.17). *Then X^1 and X^2 have the same law (i.e., for any Borel set $A \subset \Omega$, we have $\mathbb{P}(\omega : X^1(\omega) \in A) = \mathbb{P}(\omega : X^2(\omega) \in A)$).*

Remark 4.22. In [16, Theorem 2.3] it is shown that the assertion of our Theorem 4.20 holds with $|\det{}_2(I + D_H G)|$ instead of $\det{}_2(I + D_H G)$ if one assumes that

(i) $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is of class C^1;

(ii) T is bijective;

(iii) the condition of [16, Proposition 2.2] holds (such condition guarantees the validity of (4.18), for any $\omega \in \Omega$, and so it implies (4.17), for any $\omega \in \Omega$).

We point out that, in the notation of [16], $\det{}_2(I + D_H G)$ is written as $\det{}_c(-D_H G)$).

4.6 An application

Here we show an explicit stochastic boundary value problem for which uniqueness in law holds, but it seems that no known method allows to prove pathwise uniqueness (see, among others, [6] and the seminal paper [13]). For such problem we can also establish existence of solutions (see also Remark 4.24 for a more general existence theorem). The result looks similar to Theorem 3.2 (where we have proved existence and pathwise uniqueness). However, note that here *the non-resonance condition* (3.6) *can be violated in a discrete set of points.*

Theorem 4.23. *Let us consider the boundary value problem* (1.1) *with* $f(t, x, y) = f(x)$. *Assume that* $f \in C_b^2(\mathbb{R})$ *and, moreover, that*

$$\begin{aligned} &(i)\ 0 < f'(x) \le \pi^2, \quad \text{for any } x \in \mathbb{R}; \\ &(ii)\ A = \{x \in \mathbb{R} : f'(x) = \pi^2\} \ \text{is discrete.} \end{aligned} \tag{4.35}$$

Then there exists a solution X. Moreover, uniqueness in law holds for (1.1) *(i.e., any solution Z of* (1.1) *has the same law of X).*

Proof. Uniqueness. We will suitably apply Corollary 4.21. To this purpose it is enough to show that any solution X of (1.1) verifies condition (L), i.e., there exists an admissible Borel set $\Omega_0 \subset \Omega$ such that

$$(L) \begin{cases} \text{for any } \omega \in \Omega_0,\ \mathbb{P}\text{-a.s., the BVP: } u_t'' + f'(X_t(\omega))u_t = 0, \ \ u_0 = u_1 = 0, \\ \text{has only the zero solution.} \end{cases}$$

Let us consider the following set Ω_0:

$$\Omega_0 = \{\omega \in \Omega \ : \ f'(X_t(\omega)) < \pi^2, \quad t \in [0,1], \ a.e.\}.$$

By looking at $\Omega \backslash \Omega_0$, it is not difficult to prove that Ω_0 is Borel. Note that $\Omega \backslash \Omega_0$ contains all $\omega \in \Omega$ such that there exists an interval $I_\omega \subset [0,1]$ on which $t \mapsto f'(X_t(\omega)) = \pi^2$. The proof is now divided into three steps.

I Step. We show that, for any $\omega \in \Omega_0$, (L) holds.

We will use the following well-known result (it is a straightforward consequence of [6, Lemma 3.1, page 92]). Let ρ_t, $t \in [0,1]$, be a real and measurable function. Assume that there exists $h > 0$ such that $h < \rho_t < \pi^2$, $t \in [0,1]$, *a.e.*. Then the linear boundary value problem $v_t'' + \rho_t v_t = 0, \ \ v_0 = v_1 = 0$, has only the zero solution.

Let $\omega \in \Omega_0$. In order to apply the previous result, we remark that,

$$h_\omega < f'(X_t(\omega)) < \pi^2, \quad t \in [0,1], \ a.e., \tag{4.36}$$

for some $h_\omega > 0$. This follows, since $t \mapsto f'(X_t(\omega))$ is continuous and positive on $[0,1]$.

II Step. We show that $\mathbb{P}(\Omega_0) = 1$.
Take any $\omega \in \Omega \setminus \Omega_0$. There exists a time interval $I_\omega \subset [0,1]$ such that

$$\pi^2 = f'(X_t(\omega)), \quad t \in I_\omega.$$

By using the continuity of the mapping $t \mapsto f'(X_t(\omega))$ and the fact that A is discrete, we infer that there exists $x_\omega \in A$ such that $X_t(\omega) = x_\omega$, $t \in J_\omega$, for some time interval J_ω contained in I_ω. This means that

$$\int_0^1 K(t,s) f(X_s(\omega)) ds + Y_t(\omega) = x_\omega,$$

for any $t \in J_\omega$ (see Lemma 2.3). Differentiating with respect to t, we get

$$\int_0^1 \frac{\partial K}{\partial t}(t,s) f(X_s(\omega)) ds = -Y_t'(\omega) = \int_0^1 \omega_s ds - \omega_t, \quad t \in J_\omega.$$

It is well-known that the map $\xi_t(\omega) = \int_0^1 \frac{\partial K}{\partial t}(t,s) f(X_s(\omega)) ds$ belongs to $C^1([0,1])$. We have found

$$\omega_t = \int_0^1 \omega_s ds - \xi_t(\omega), \quad t \in J_\omega.$$

On the right hand side, we have a function which is C^1 on J_ω. This means that, for any $\omega \in \Omega \setminus \Omega_0$, there exists a time interval on which ω is a C^1-function. Since the Wiener process (see (1.3)), $\mathbb{P}$-a.s., has trajectories which are never of bounded variation in any time interval of $[0,1]$, we have that $\mathbb{P}(\Omega \setminus \Omega_0) = 0$.

III Step. We prove that, for any $\omega \in \Omega_0$, $\mathbb{P}$-a.s., we have $\omega + H_0 \subset \Omega_0$.

Assume by contradiction that this is not true. This means that, there exists a Borel set $\Omega' \subset \Omega_0$ with $\mathbb{P}(\Omega') > 0$, such that, for any $\omega \in \Omega'$ there exists $h \in H_0$ with $\omega + h \notin \Omega_0$. Let us consider such ω and h.
Arguing as before, we find that there exists a time interval $J_{\omega+h} \subset [0,1]$ and some $x_{\omega+h} \in A$ such that $X_t(\omega + h) = x_{\omega+h}$, $t \in J_{\omega+h}$. This means that

$$\omega_t + h_t = \int_0^1 \omega_s ds + \int_0^1 h_s ds - \xi_t(\omega + h), \quad t \in J_{\omega+h}.$$

We have found that for each $\omega \in \Omega'$ there exists a time interval on which ω is of bounded variation. This contradicts the fact that $\mathbb{P}(\Omega') > 0$ and finishes the proof of uniqueness.

<u>*Existence.*</u> The proof is divided into three steps.

I Step. For any $\omega \in \Omega$, consider the sequence $(X^n(\omega))$, with $X_t^1(\omega) = 0$, $t \in [0,1]$, and

$$X_t^{n+1}(\omega) = \int_0^1 K(t,s) f(X_s^n(\omega)) ds + Y_t(\omega), \quad n \geq 1, \quad t \in [0,1].$$

Using the boundedness of f, an application of the Ascoli-Arzelà theorem shows that, for any $\omega \in \Omega$, there exists a subsequence $(X^k(\omega))$ (possibly depending on ω) which converges in $C([0,1])$ to a continuous function $X(\omega)$. It is then clear that, for any $\omega \in \Omega$, we have

$$X_t(\omega) = \int_0^1 K(t,s) f(X_s(\omega)) ds + Y_t(\omega), \quad t \in [0,1]. \tag{4.37}$$

The main difficulty is that the previous construction does not clarify the measurable dependence of X on ω. To this purpose we will suitably modify X in order to obtain the required measurability property.

II Step. We investigate when condition (LY) in (4.18) holds, i.e., for which $\omega \in \Omega$

$$\begin{cases} \text{the linearized BVP: } u_t'' + f'(Y_t(\omega))u_t = 0, \;\; u_0 = u_1 = 0, \\ \text{has only the zero solution.} \end{cases} \tag{4.38}$$

Arguing as in the proof of uniqueness, condition (4.38) holds in particular if ω satisfies

$$h_\omega < f'(Y_t(\omega)) < \pi^2, \quad t \in [0,1], \; a.e., \tag{4.39}$$

for some $h_\omega > 0$. On the other hand, if (4.39) does not hold for $\omega^0 \in \Omega$, then there exists $x_{\omega^0} \in A$ such that $Y_t(\omega^0) = x_{\omega^0}$, $t \in J_{\omega^0}$, for some time interval $J_{\omega^0} \subset [0,1]$. It follows that $Y_t(\omega^0) = x_{\omega^0}$, for any $t \in J_{\omega^0}$. Differentiating with respect to t, we get

$$0 = -\int_0^1 \omega_s^0 ds + \omega_t^0, \quad t \in J_{\omega^0}.$$

This implies that $\omega_t^0 = \int_0^1 \omega_s^0 ds$, $t \in J_{\omega^0}$. Let us introduce the set $\Lambda \subset \Omega$ of all ω such that there exists a time interval $I_\omega \subset [0,1]$ on which ω is a function of bounded variation. It is not difficult to prove that Λ is a Borel subset of Ω. Moreover, $\mathbb{P}(\Lambda) = 0$.

We have just verified that (4.38) holds for any $\omega \in \Omega \setminus \Lambda$.

III Step. Let us consider the mapping $X(\omega)$ of Step I and introduce $S : \Omega \to \Omega$,

$$S_t(\omega) = \omega_t - \int_0^t f(X_s(\omega))ds.$$

We have $X(\omega) = Y(S(\omega))$ and $T(S(\omega)) = \omega$, for any $\omega \in \Omega$ as in Section 4.3. Although *S is not necessarily measurable,* one can easily check that

$$S^{-1}(\Lambda) = \Lambda.$$

This implies that $S(\Omega \setminus \Lambda) = \Omega \setminus \Lambda$ (clearly $\mathbb{P}(\Omega \setminus \Lambda) = 1$). Now we argue as in the proof of Theorem 4.14 with its notations. Since we know that (4.38) is verified when $\omega = S(\theta)$, for some $\theta \in \Omega \setminus \Lambda$, we deduce that the Fréchet derivative $DT(S(\omega))$ is an isomorphism from Ω into Ω, for any $\omega \in \Omega \setminus \Lambda$.

By the inverse function theorem, T is a local diffeomorphism from an open neighborhood $U_{S(\omega)}$ of $S(\omega)$ to an open neighborhood $V_{T(S(\omega))} = V_\omega$ of $T(S(\omega)) = \omega$, for any $\omega \in \Omega \setminus \Lambda$. Let us denote by T^{-1} the local inverse function. We deduce that, for any $\omega \in \Omega \setminus \Lambda$, $S(\theta) = T^{-1}(\theta), \quad \theta \in V_\omega$.

Introduce the open set

$$\Phi = \bigcup_{\omega \in \Omega \setminus \Lambda} V_\omega.$$

Since $\Omega \setminus \Lambda \subset \Phi$, we have that $\mathbb{P}(\Phi) = 1$. In addition Φ is an admissible open set in Ω, since, for any $\omega \in \Omega \setminus \Lambda$, we have that $\omega + H_0 \subset \Omega \setminus \Lambda \subset \Phi$.

The restriction of S to Φ is a C^1-function with values in Ω. We define the measurable mapping

$$\hat{S} : \Omega \to \Omega, \quad \hat{S}(\omega) = \begin{cases} S(\omega), & \omega \in \Phi \\ 0, & \omega \in \Omega \setminus \Phi \end{cases}$$

and introduce $\hat{X} : \Omega \to \Omega$, $\hat{X}_t(\omega) = Y_t(\hat{S}(\omega))$, $\quad \omega \in \Omega, \ \ t \in [0,1]$.
It is clear that $\hat{X}$ is measurable. Moreover, since $\hat{X}(\omega) = X(\omega)$, when $\omega \in \Phi$, we have that $\hat{X}$ verifies (4.37) for any $\omega \in \Phi$. This shows that $\hat{X}$ is a solution to (1.1) and finishes the proof. $\square$

An example of f which is covered by the previous result is

$$f(x) = \pi^2 \int_0^x e^{-t^2} dt, \quad x \in \mathbb{R}.$$

Remark 4.24. The previous proof shows that an existence result for (1.1) holds, more generally, if the following three conditions hold:
(i) $\ f(t,x,y) = f(x)$ with $f \in C_b(\mathbb{R}) \cap C^1(\mathbb{R})$;
(ii) there exists a Borel set $\Lambda \subset \Omega$ such that $\Omega \setminus \Lambda$ is admissible and, moreover, $S(\Omega \setminus \Lambda) \subset \Omega \setminus \Lambda$, where $S : \Omega \to \Omega$ is defined by

$$S_t(\omega) = \omega_t - \int_0^t f(Z_s(\omega))ds, \quad \omega \in \Omega, \ \ t \in [0,1],$$

where $Z : \Omega \to \Omega$ is any mapping (non necessarily measurable);
(iii) condition (4.38) holds, for any $\omega \in \Omega \setminus \Lambda$.
Under (i)-(iii), the existence of solution can be proved by adapting the proof of Theorem 4.23.

5 Remarks on computation of the Carleman-Fredholm determinant $\det_2(I + D_H G)$

When dealing with non-adapted versions of the Girsanov theorem (see [12], [20][25]) one delicate problem is to find some explicit expression for the Carleman-Fredholm determinant appearing also in (4.12) of Section 4.2. This problem has been also considered in [7], [8], [16] and [26] for different measurable transformations $\mathcal{T}$. In particular the Radon-Nykodim derivative appearing in Theorem 4.20 and [16, Theorem 2.3] (see also Remark 4.22) contains the explicit term

$$\det{}_2(I + D_H G(\omega))$$

(in the notation of [16], $\det{}_2(I + D_H G(\omega))$ becomes $\det{}_c(-D_H G(\omega))$).
The assertion in our next result is a reformulation of [16, Lemma 2.4]. It provides an explicit formula for $\det{}_2(I+D_H G(\omega))$. It is important to point out that our computation of the Carleman-Fredholm determinant $\det_2(I + D_H G(\omega))$ has been developed with techniques which are completely different from those (based on Malliavin calculus) used for the proof of Lemma 2.4 in [16].
Our approach comes from [10] and it uses functional analysis and the theory of linear ordinary differential equations. For the reader's convenience, we have collected in Appendix B some of the ideas (taken from [10]) which have enabled us to perform our computation of the Carleman-Fredholm determinant and some important consequences of this approach.
We believe that this method could be useful in other situations (cf. [2], [7], [8], [26]).

Lemma 5.1. *Assume that $f \in C^1$ and that the linearized BVP*

$$u_t'' + b_t(\omega)u_t' + a_t(\omega)u_t = 0, \quad u_0 = u_1 = 0,$$

where $a_t = f_x(t, Y_t(\omega), Y_t'(\omega))$, $b_t = f_y(t, Y_t(\omega), Y_t'(\omega))$, has the only zero solution, for any $\omega \in \Omega$.
Then the following relation holds

$$\det{}_2(I + D_H G(\omega)) = Z_1(\omega) \exp\Big(\int_0^1 (ta_t + (1-t)b_t)dt \Big),$$

where Z_t solves the Cauchy problem

$$u_t'' + b_t u_t' + a_t u_t = 0, \quad u_0 = 0, \quad u_0' = 1.$$

Proof. The proof is based on some ideas which are developed in Appendix B. More precisely, observe that, by (5.3), the assumption in Lemma 5.1 guarantees that we can apply Theorem 5.4 with $L = D_H G(\omega)$. □

References

[1] A. Arciniega, Shooting methods for numerical solution of nonlinear stochastic boundary-value problems. *Stoch. Anal. Appl.* **25** (2007), 187-200.

[2] A. Alabert and D. Nualart, A second order Stratonovich differential equation with boundary conditions, *Stochastic Processes and their applications* **68** (1997), 21-47.

[3] P. B. Bailey, L. F. Shampine and P. E. Waltman, *Nonlinear two-point boundary value problems*, Academic Press, 1968.

[4] H. Brézis, *Analyse fonctionnelle. Théorie et applications*, Masson,1983.

[5] R. Buckdahn, Anticipative Girsanov transformations and Skorohod stochastic differential equations, *Seminarbericht* **92-2** (1992).

[6] S.N. Chow and J.K. Hale, *Methods of bifurcation theory*, Springer-Verlag, 1982.

[7] C. Donati-Martin, Quasi-linear elliptic stochastic partial differential equation: Markov property, *Stochastics Stochastics Rep.* **41** (1992), 219-240.

[8] C. Donati-Martin and D. Nualart, Markov property for elliptic stochastic partial differential equations, *Stochastics Stochastics Rep.* **46** (1994), 107-115.

[9] E. Getzler, Degree Theory for Wiener Maps, *J. Funct. Anal.* **68** (1986), 388-403.

[10] I. Gohberg, S. Goldberg and N. Krupnik, *Traces and determinants of linear operators*, Birkhäuser Verlag, 2000.

[11] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes,* North Holland-Kodansha, 1981.

[12] S. Kusuoka, The nonlinear transformation of Gaussian measure on Banach space and absolute continuity. I., *J. Fac. Sci. Univ. Tokyo* Sect. IA Math. **29** (1982), 567-597.

[13] J. Mawhin and J. R. Jr. Ward, Nonresonance and existence for nonlinear elliptic boundary value problems, *Nonlinear Anal.* **5** (1981), 677-684.

[14] D. Nualart, *The Malliavin Calculus and related topics*, Springer-Verlag, 2006.

[15] D. Nualart and E. Pardoux, Stochastic calculus with anticipating integrands, *Prob. Theory Rel. Fields* **78** (1988), 535-581.

[16] D. Nualart and E. Pardoux, Second order stochastic differential equations with Dirichlet boundary conditions, *Stochatic Processes and Applications* **39** (1991), 1–24.

[17] D. Nualart and E. Pardoux, Boundary value problems for stochastic differential equations, *Annals of Probability* **19** (1991), 1118-1144.

[18] D. Ocone and E. Pardoux, Linear stochastic differential equations with boundary conditions, *Probab. Theory Related Fields* 82 (1989), 489-526.

[19] K. R. Parthasarathy, *Probability Measures on Metric Spaces.*, Academic Press, 1967.

[20] R. Ramer, On nonlinear transformations of Gaussian measures, *J. Functional Anal.* **15** (1974), 166-187.

[21] M. Reed and B. Simon, *Methods of modern mathematical physics. I. Functional analysis*, Academic Press, 1980.

[22] H. Sugita, On a characterization of the Sobolev spaces over an abstract Wiener space, *J. Math Kyoto Univ.* **25-4** (1985) 717-725.

[23] A. S. Üstünel, Some exponential Moment Inequalities for the Wiener functionals, *J. Funct. Anal.* **136** (1996), 154-170.

[24] A. S. Üstünel and M. Zakai, Transformation of Wiener measure under anticipative flows, *Probab. Theory Rel. Fields* **93** (1992), 91-136.

[25] A. S. Üstünel and M. Zakai, Applications of the degree theorem to absolute continuity on Wiener space, *Probab. Theory and Related Fields* **95** (1993), 509-520.

[26] A. S. Üstünel and M. Zakai, *Tranformation of Measure on Wiener space,* Springer-Verlag, 2000.

Appendix A: Some definitions of Malliavin Calculus

Here we summarize some notions of Malliavin Calculus (see [14], [11, Chapter V] and [26, Appendix B].
If K and M are real separable Hilbert spaces, we consider the tensor product of K and M, i.e. $K \otimes M$ (this is the real Hilbert space formed by all Hilbert-Schmidt operators from K into M; see [21, Chapter VI and (48) in page 220]). We also use the notation $\mathcal{HS}(K, M)$ for $K \otimes M$.
Moreover, for $k \in K$ and $h \in M$, we consider the linear operator $k \otimes h$ from K into M:

$$(k \otimes h)(u) := \langle k, u \rangle_K h, \quad u \in K.$$

Let H_0 be the Hilbert space introduced at the end of Section 1.

The smooth K-valued functionals on $(\Omega, H_0, \mathbb{P})$ are functionals $a : \Omega \to K$ of the form

$$a(\omega) = \sum_{i=1}^{N} f_i(\langle h_1, \omega \rangle, \ldots, \langle h_m, \omega \rangle) k_i,$$

where $f_i \in C_b^\infty(\mathbb{R}^m)$, $h_1, \ldots, h_m \in H_0$, $k_1, \ldots, k_N \in K$, and we set

$$\langle h, \omega \rangle = \int_0^1 h'_s d\omega_s \quad \text{(Itô integral)}, \quad h \in H_0, \;\; h' = \frac{dh}{ds}.$$

For smooth K-valued functionals we define the Malliavin derivative

$$D_M a(\omega) = \sum_{i=1}^{N}\sum_{j=1}^{m} \partial_{x_j} f_i(\langle h_1, \omega\rangle, \dots, \langle h_m, \omega\rangle) h_j \otimes k_i.$$

The Sobolev space $D^{1,2}(K) \subset L^2(\Omega, K)$ is now the completion of smooth K-valued functionals with respect to the norm

$$\|a\|_{1,2} = \|a\|_{L^2(\Omega,K)} + \|D_M a\|_{L^2(\Omega;H_0\otimes K)}.$$

The Malliavin derivative on $D^{1,2}(K)$ (still denoted by D_M) is the closure of D_M as defined on smooth K-valued functionals.
When $K = \mathbb{R}$, the adjoint of D_M is denoted by δ and is called the Skorohod integral. Hence if $\xi \in L^2(\Omega, H_0)$, we say that $\xi \in \mathrm{dom}(\delta)$ if we have

$$\mathbb{E}[\langle D_M \phi, \xi\rangle_{H_0}] \le \|\phi\|_{L^2}\, c(\xi),$$

for any $\phi \in D^{1,2}(\mathbb{R})$. If $\xi \in \mathrm{dom}(\delta)$, we have $\delta\xi \in L^2(\Omega, \mathbb{R})$ and

$$\mathbb{E}[\langle D_M \phi, \xi\rangle_{H_0}] = \mathbb{E}[\phi\, \delta(\xi)].$$

We also need to introduce the second Malliavin derivative. Let $F : \Omega \to H_0$ be a measurable mapping which belongs to $D^{1,2}(H_0)$. If $D_M F \in D^{1,2}(H_0 \otimes H_0)$ then we say that $F \in D^{2,2}(H_0)$ and set $D_M^2 F = D_M(D_M F)$. Note that, for any ω, $\mathbb{P}$-a.s.,

$$D_M^2 F(\omega) \in \big(H_0 \otimes (H_0 \otimes H_0)\big) = \mathcal{HS}(H_0, H_0 \otimes H_0).$$

Appendix B: An input-output representation for linear boundary value problems

In this section, we briefly sketch the framework of [10, Chapter XIII] in which our computation of the Carleman-Fredholm determinant (Lemma 5.1) is developed. Throughout this section, since only deterministic functions are involved, we go back to the notation $\alpha(t) = \alpha_t$, for any real function α. We are concerned with the Hilbert-Schmidt integral operator defined as follows:

$$(Lh)(t) = -a(t)\int_0^1 K(t,s)h(s)ds - b(t)\int_0^1 \frac{\partial K}{\partial t}(t,s)h(s)ds, \quad h \in H,\ t \in [0,1]. \tag{5.1}$$

Here a and b are given real continuous functions on $[0,1]$ and

$$K(t,s) = t \wedge s - ts$$

is the Green's function of $-d^2/dt^2$ (with Dirichlet boundary condition).
From the definition of the integral operator L in (5.1), it is easy to check that for any $y \in H$ a function ξ solves

$$\xi''(t) + b(t)\xi'(t) + a(t)\xi(t) = y(t), \quad \xi(0) = \xi(1) = 0 \tag{5.2}$$

if and only if, setting $u := \xi''$, it is $(I + L)u = y$. In other words,

$$\text{Problem (5.2) is solvable} \iff \text{the operator } (I+L) : H \to H \text{ is invertible.} \tag{5.3}$$

Note that the equation in (5.2) can be rewritten as

$$\begin{cases} u = \xi'' \\ y = a(t)\xi + b(t)\xi' + u, \\ \xi(0) = \xi(1) = 0. \end{cases} \tag{5.4}$$

In [10, Section XIII], (5.4) is called an input-output representation of $(I + L)$, where u is the input and y is the output. More precisely, setting $\xi = x^1$, $\xi' = x^2$, (5.4) is of the form

$$\begin{cases} x' = Ax + Bu \\ y = C(t)x + u \\ N_1 x(0) + N_2 x(1) = 0, \end{cases} \tag{5.5}$$

with

$$N_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ N_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, C(t) = (a(t), b(t)).$$

It can be easily verified that the inverse $u = (I + L)^{-1}y$ admits the following representation

$$\begin{cases} x' = (A - BC(t))x + By \\ u = -C(t)x + y \\ N_1 x(0) + N_2 x(1) = 0, \end{cases} \tag{5.6}$$

i.e.

$$\begin{cases} \xi'' = -a(t)\xi - b(t)\xi' + y \\ u = -a(t)\xi - b(t)\xi' + y, \\ \xi(0) = \xi(1) = 0. \end{cases} \tag{5.7}$$

We now introduce the fundamental matrices $U^\times$,

$$\frac{dU^\times}{dt}(t) = (A - BC(t))U^\times(t), \quad U^\times(0) = I, \quad i.e., \quad U^\times(t) = \begin{pmatrix} u_1(t) & u_2(t) \\ u_1'(t) & u_2'(t) \end{pmatrix}, \tag{5.8}$$

where $u_k'' + b(t)u_k' + a(t)u_k = 0$, $k = 1, 2$, $u_1(0) = u_2'(0) = 1$, $u_1'(0) = u_2(0) = 0$, and

$$U(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

With the previous notation, one can prove

Proposition 5.2. *Assume that $(I + L)$ is invertible. Then there exists a positive constant C (depending only on the coefficients a and b through their supremum norms $\|a\|_0$ and $\|b\|_0$) such that*

$$|(I + L)^{-1}y|_H \le C|y|_H, \quad y \in H.$$

Proof. We make straightforward estimates on the control problem (5.6) based on the Gronwall lemma. □

One can also deduce from [10, Theorem XIII.5.1] the next result, which leads to Proposition 4.16 in Section 4.

Theorem 5.3. *With the previous notation, assume that* $(I+L)$ *is invertible. Then*

$$u(t) = (I+L)^{-1}y(t) = y(t) - \int_0^1 \gamma(t,s)y(s)ds, \quad t \in [0,1], \tag{5.9}$$

where with $P^\times = (N_1 + N_2U^\times(1))^{-1}N_2U^\times(1)$,

$$\gamma(t,s) = \begin{cases} C_tU^\times(t)(I-P^\times)U^\times(s)^{-1}B, & 0 \le s < t \le 1, \\ -C_tU^\times(t)P^\times U^\times(s)^{-1}B, & 0 \le t < s \le 1, \end{cases}$$

or, more explicitly,

$$\gamma(t,s) = \begin{cases} (\frac{1}{W})[a(t)u_2(s)\psi(t) + b(t)u_2(s)\psi'(t)], & 0 \le s < t \le 1, \\ (\frac{1}{W})[a(t)u_2(t) + b(t)u_2'(t)]\varphi(s), & 0 \le t < s \le 1, \end{cases}$$

where u_1 *and* u_2 *are introduced in* (5.8), $W = u_1u_2' - u_2u_1'$, $M = u_1(1)/u_2(1)$ *and*

$$\varphi(s) = -u_2(s)M + u_1(s), \ \psi(t) = u_2(t)M - u_1(t), \quad t \in [0,1], \quad s \in [0,1].$$

Finally by [10, Theorem XIII.7.1], we obtain

Theorem 5.4. *Assume that* $(I+L)$ *is invertible. Setting*

$$P = (N_1 + N_2U(1))^{-1}N_2U(1),$$

we have

$$\begin{aligned} \det{}_2(I+L) =& \det(I - P + PU(1)^{-1}\,U^\times(1))\, e^{\int_0^1 \operatorname{tr}(C_tU(s)PU^{-1}(s)B)ds} \\ =& u_2(1)\, \exp\Big(\int_0^1 (ta(t)+b(t))(1-t)dt\Big), \end{aligned}$$

where $u_2'' + b_tu_2' + a_tu_2 = 0$, $u_2'(0) = 1$, $u_2(0) = 0$.